DOCUMENT RESUME

ED 251 303 SE 045 253

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TITLE The Negative Numbers. A Source-Work Collection for In-Service and Pre-Service Teacher Courses.
INSTITUTION Weizmann Inst. of Science, Rehovot (Israel).
PUB DATE Dec 83
NOTE 156p.; For a related document, see SE 045 252.
AVAILABLE FROM Weizmann Institute, Department of Science Teaching, 71600 Rehovot, Israel. Single copies for academic use are free; others $6.00/set.
PUB TYPE Guides - Classroom Use - Materials (For Learner) (051) -- Guides - Classroom Use - Guides (For Teachers) (052)

EDRS PRICE MF01 Plus Postage. PC Not Available from EDRS.
DESCRIPTORS *History; Inservice Teacher Education; *Instructional Materials; Integers; Junior High Schools; *Mathematics Education; *Mathematics Instruction; Number Concepts; Preservice Teacher Education; Resource Materials; *Teacher Education; Teaching Methods; Worksheets; Workshops
IDENTIFIERS Israel; *Negative Numbers

ABSTRACT

These materials on negative numbers are designed for use in workshops for pre- and in-service teacher training, especially for teachers in junior high schools. The general aims are: to improve teachers' mathematical knowledge of topics, to enrich teachers' mathematical background, to allow opportunity to discuss relevant didactics, and to create a reasonable image of mathematics and mathematical activity as a human endeavor. The worksheets (in booklet form) generally have a brief biographical-chronological introduction to set the historical scene, an historical source, and leading questions on the source materials and on mathematical and didactical consequences thereof. Each worksheet is accompanied by answer sheets (also in booklet form), containing solutions and further sources and information. Eleven worksheets present the history of negative numbers under such topics as: F. Viete, R. Descartes, contradictions in the use of negative numbers, N. Saunderson, L. Euler, W. Frend, G. Peacock, and the formal entry of negative numbers into mathematics. Comments on how to use the worksheets are briefly presented. (MNS)

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THE NEGATIVE NUMBERS

A source-work collection for in-service and pre-service teacher courses

A. Arcavi

M. Bruckheimer

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Introduction and commentaries to the set of worksheets on the historical development of the negative numbers

The attached materials are designed for use in workshops for pre and in-service teacher training, in particular for teachers in junior high schools. Although the sheets have a single topic, viz. the negative numbers, many general points concerning mathematical activity and its teaching are included (see the individual commentaries).

Thus the general aims of this and other sequences of worksheets is:

1- to *improve* the teachers' *mathematical knowledge* of topics included in the curriculum, and that in a way which motivates the teacher to reconsider topics previously studied but imperfectly understood;

2- to *enrich* the teachers' *mathematical background* to topics in the curriculum;

3- to allow opportunity for the *discussion of relevant didactics* and to consider them in relation to the pure mathematics of the topics concerned;

4- to create a *reasonable image of mathematics* and mathematical activity as a human endeavour. In particular, to create an awareness of the history of topics included in the curriculum.

In general, the worksheets in this sequence have the following form:

- a brief biographical-chronological introduction in order to set the historical scene,
- an historical source; as far as possible a primary source,
- leading questions on the source material and on mathematical and didactical consequences thereof.

To each worksheet an extensive discussion of the solutions, and points arising from them, is given in the respective *answer sheet*. Thus, the answer sheets contain not only the detailed solutions to the questions, but further source material, background, historical and mathematical information. Both the worksheets and the answer sheets are designed as learning materials.

At the present time, this sequence contains 11 worksheets which are intended to be worked in the following order:

1- Introduction.
2- F. Viète.
3- R. Descartes.
4- Contradictions in the use of negative numbers.
5- N. Saunderson.
6- L. Euler.
7- The last of the opposition - W. Frend.
8- G. Peacock.
9- The formal entry of the negative numbers into mathematics. Part I: Definitions.
10- The formal entry of the negative numbers into mathematics. Part II: Proofs. Part III: And so on...
11- Summary.

We have used the sheets in the following:

i- In-service workshops.
ii- Pre-service courses.
iii- Correspondence course.
iv- Single worksheets in other workshops.

In each case the work is guided by a tutor. Student-teachers were given the first worksheet. They worked in groups or individually, with the tutor's "interference" if necessary. Then a collective guided discussion took place, and the answersheet was distributed. And so on.
For use with Israeli teachers, we translated the sources freely into Hebrew and gave the original with the translation, in order to encourage the student to read both (if he understood the original language) and also that the original should be available in cases of misunderstanding attributable to mistranslation. If this sequence is to be used with non-English speaking students, it would seem desirable to do likewise. In the present version, there are some extracts in languages other than English, and so even English users have to consider whether to translate them or not.

This series is not a text. Therefore there is a need for the tutor using the sheets, to add comments, to be in a position to answer questions, and to have in mind the overall structure of the set.

In the following commentary we give briefly, some of the points we had in mind when writing and using the worksheets.

Commentary on the individual worksheets

1- Introduction

In contrast to other worksheets, the introduction is:

- not a *work*sheet, but is intended to be read,
- not a primary source, but a secondary source.

The object is to summarize the history of negatives, thus providing a simple framework in which to work. It also gives information on some early stages in the development of negative numbers, for which we do not bring worksheets. Note that the sentence concerning Viète is erased, because extracts from Viète are the subject of the following worksheet.

2- F. Viète

This worksheet exemplifies the complete avoidance of negative numbers. One of the points being made here is that although the term "negative" and the law of signs appear in the text, the negative numbers as such, are not recognized. The didactics are stressed, since the formulae are relevant to the junior high school curriculum.

3- R. Descartes

The extract exemplifies the use of negatives arising as roots of equations but still not having the "status" of numbers (*racines fausses, le défault d'une quantité*) We exploit the opportunity to do a little mathematics concerning the roots of polynomials and to prove or justify intuitive or simply stated results in the text.

4- Contradictions in the use of negative numbers

Two contradictions which arose from the use of negative numbers are presented. The student is expected to deal with them using all the mathematics at his disposal. The answer sheet expands on the solution by bringing also the responses of mathematicians of the time. An important point is the general concept of extending properties from one set to a larger set. This can be considered as a prelude to the worksheet on Peacock.

5-6 N. Saunderson - L. Euler

These two worksheets are examples of didactics in the 18th century, which was characterized by the large number of algebra textbooks, in which the authors dealt freely with negatives, and attempted to explain their use.

The purpose is to compare different presentations of the multiplication of negatives, and to emphasize the "problem" of giving a satisfactory explanation when the negatives themselves are not mathematically defined.

7- The last of the opposition - W. Frend

The diminishing, but still present, opposition to the use of negatives at the begining of the 19th century, is exemplified.

The extracts cover:

-arguments against negatives,

-a way of dealing with quadratic equations avoiding negatives completely.

The answer sheet brings extracts from A.N. Whitehead, A. De Morgan and others who, beside countering the rejectionists' arguments, also make general points on the nature of mathematics.

8- G. Peacock

Peacock made an attempt at formalization in answer to the rejectionists. A discussion with many examples of the didactical implications of the *Principle of Permanence of Equivalent Forms* is brought.

The answer sheet is enriched by extracts from B. Russell, F. Klein and others.

9-10 The formal entry of the negative numbers into mathematics

The previous worksheets are an illustration of the following assertion: "The history of mathematics is a bountiful source of ... examples showing ways in which previous generations have experimented and discovered the need for formal mathematical structures".(Meserve*) So the end of the story must be a way of introducing the whole numbers formally. We do it by means of equivalence classes of ordered pairs.

These worksheets are not based on historical sources. They are developed by means of a carefully structured sequence of leading questions. The reason for the change of style is that we have not found a satisfactory source on which to base the formal development, and without this stage the story is incomplete.

After working these two sheets we recommend to our students that they re-read the introductory sheet.

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* Meserve B., The History of Mathematics as a Pedagogical Tool. In Zweng M. (ed), Proceedings of the 4th ICME Birkhauser, 1983, p. 398-400.

11- Summary

The sheet consists of two parts. The first brings an extract from a secondary source which summarizes certain periods in the history of the negative numbers, citing mathematicians, books and arguments that the student has met in his work on the sequence.

The objective is to reinforce details from previous sheets. The opportunity is also taken to point out the problem of obtaining reliable historical information.

The second part consists of a chronological table and a suggested division of the development of negatives into stages. As a final exercise the student is asked to correlate the mathematicians and the stages and to fit both into the chronological table.

Final Comment

We do not see these sheets as final or definitive. As we find new and "better" sources, or in the light of experience in the use of the sheets, we make alterations, corrections, replacements, additions, etc.

Other sequences are also in process of development, both with the intention of improving the development per se, and evaluating their implementation.

A. Arcavi
M. Bruckheimer

December 1983

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The Negative Numbers

Introduction

The following quote is taken from the book by E.T. Bell, The Development of Mathematics. It summarises a part of the history of the negative numbers.

Professional historians are in substantial agreement on the following details in the development of negatives. Diophantus, in the third century A.D., encountering -4 as the formal solution of a linear equation, rejected it as absurd. In the first third of the seventh century, Brahmagupta is said to have stated the rules of signs in multiplication; he discarded a negative root of a quadratic. The rules of signs became common in India after their restatement by Mahavira in the ninth century. Al-Khowarizmi, of about the same time, made no advance except that he appears to have exhibited a positive and a negative root for a quadratic without explicitly rejecting the negative.

Of the Europeans, Fibonacci in the early thirteenth century rejected negative roots, but took a step forward when he interpreted a negative number in a problem concerning money as a loss instead of a gain. It has been claimed that the Indians did likewise. L. Pacioli (1445?–1514, Tuscan) in the second half of the fifteenth century is credited with a knowledge of the rule of signs on such evidence as $(7 - 4)(4 - 2) = 3 \times 2 = 6$. M. Stifel[1] (1487?–1567, German) a fine algebraist for his time, called negative numbers absurd in the middle of the sixteenth century. Cardan, in his *Ars magna* (1545), stated the rule 'minus times minus gives plus' as an independent proposition; he also is said to have recognized negative numbers as 'existent,' but on evidence which seems doubtful. In fact, he called negatives 'fictitious.'

Bombelli in 1572 showed that he understood the rules of addition in such instances as $m - n$, where m, n are positive integers. Vieta, about the same time,...
Finally J. Hudde (1628–1704, Dutch) in 1659 used a letter to denote a positive or a negative number indifferently. As an historical curiosity, it may be mentioned that T. Harriot (1560–1621, English) was one of the first Europeans to duplicate the feat of the ancient Babylonians in permitting a negative number to function as one member of an equation. But he refused to admit negative roots.

With one exception, all items in the foregoing list may be classified as partial extension by formalism. The extension was incomplete because no free use of negatives was made until the seventeenth century. The extension was formal because it had no basis other than the mechanical application of rules of calculation that were known to produce consistent results when applied to positive numbers, and were assumed to be legitimate in the manipulation of negatives. This unbased assumption was to be elevated in the 1830's to the dignity of a general dogma in the notorious and discredited 'principle of permanence of form.'

By the middle of the seventeenth century, untrammeled use of negatives had given mathematicians a pragmatic demonstration that the rules of common algebra lead to consistent results. But there was no attempt to go any deeper and put a substratum of postulates under the rickety formalism.

The one glimmer of mathematical intelligence in the early history of negatives is the suggestion of Fibonacci that a negative sum of money may be interpreted as a loss. This appears to have been the first step toward the second stage in the evolution of negatives, that of interpreting the results of formalism in terms of something which is accepted as consistent. It marks the beginning of two distinct but complementary philosophies of mathematics: the products of mathematical formalism are to be admitted only if they can be put in correspondence with some already established system accepted as self-consistent; all mathematics is a formalism without meaning beyond that implied by the postulates defining the formalism. For example, if Euclidean geometry is accepted as self-consistent, and if the formal algebraic operations with complex numbers can be interpreted in terms of that geometry, the formalism of complex numbers is admissible. This is according to the first philosophy, which was that instinctively and subconsciously adopted by Fibonacci in his encounter with negatives. The second philosophy is illustrated by the rules of algebra in any modern elementary text, where $a, b, c, \cdots, +, \times, =$ are displayed, and it is postulated that $a = a$, $a + b = b + a$, etc.

Each philosophy has greatly enriched mathematics. The first, seeking interpretations, may be called synthetic; the second, beginning and ending in a formalism within its own postulated universe,[2] may be termed analytic.

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The Negative Numbers

Worksheet: Francois Viète

Francois Viète (or Franciscus Vieta, in the Latin spelling), a Frenchman born in 1540 and died in 1603. Viète studied law and most of his life worked in the public service, devoting his spare time to mathematics, in which he made his name. He wrote books on algebra and geometry. His important work, *In Artem Analyticem Isagoge (Introduction to the Analytic Art)*, appeared in 1591.

One of the innovations attributed to Viète is the use of letters, not only for variables, but also for coefficients. He thus made it possible to deal with general algebraic forms. Even though the notation in his work is much more complicated than that of today, his contribution to the advance of algebra is highly rated and he has been dubbed the "father of modern algebra".

The following is a quotation from the above book. The text is taken from the English translation brought by Klein (1968)*.

Details on the man and his contribution to mathematics can be found in

* Klein J., *Greek Mathematical Thought and the Origin of Algebra*
The M.I.T. Press, 1968.

Precept II

To subtract a magnitude from a magnitude

Let there be two magnitudes *A* and *B*, and let the former be greater than the latter. It is required to subtract the less from the greater. ... subtraction may be fittingly effected by means of the sign of the disjoining or removal[24] of the less from the greater; and disjoined, they will be *A* "minus" *B*,...

Nor will it be done differently if the magnitude which is subtracted is itself conjoined with some magnitude, since the whole and the parts are not to be judged by separate laws; thus, if "*B* 'plus' *D*" is to be subtracted from *A*, the remainder will be "*A* 'minus' *B*, 'minus' *D*," the magnitudes *B* and *D* having been subtracted one by one.

But if *D* is already subtracted from *B* and "*B* 'minus' *D*" is to be subtracted from *A*, the result will be "*A* 'minus' *B* 'plus' *D*," because in the subtraction of the whole magnitude *B* that which is subtracted exceeds by the magnitude *D* what was to have been subtracted. Therefore, it must be made up by the addition of that magnitude *D*.

The analysts, however, are accustomed to indicate the performance of the removal by means of the symbol —. ...

But when it is not said which magnitude is greater or less, and yet the subtraction must be made, the sign of the difference is: =, i.e., when the less is undetermined; as, if "*A* square" and "*B* plane" are the proposed magnitudes, the difference will be: "*A* square = *B* plane," or "*B* plane = *A* square."

Precept III.

To multiply a magnitude by a magnitude

Let there be two magnitudes *A* and *B*. It is required to multiply the one by the other. ... their product will rightly be designated by the word "*in*" or "*sub*," as, for example, "*A* in *B*," by which it will be signified that the one has been multiplied by the other...

If, however, the magnitudes to be multiplied, or one of them, be of two or more names, nothing different happens in the operation.[27] Since the whole is equal to its parts, therefore also the products under the segments of some magnitude are equal to the product under the whole. And when the positive name[28] (nomen adfirmatum) of a magnitude is multiplied by a name also positive of another magnitude, the product will be positive, and when it is multiplied by a negative name (nomen negatum), the product will be negative.

From which precept it also follows that by the multiplication of negative names by each other a positive product is produced, as when "$A=B$" is multiplied by "$D=G$" [redacted] since the product of the positive *A* and the negative *G* is negative, which means that too much is removed or taken away, inasmuch as *A* is, inaccurately, brought forward (producta) as a magnitude to be multiplied [redacted] and since, similarly, the product of the negative *B* and the positive *D* is negative, which again means that too much is removed, inasmuch as *D* is, inaccurately, brought forward as a magnitude to be multiplied [redacted] Therefore, by way of compensation, when the negative *B* is multiplied by the negative *G*, the product is positive.

Questions

1. Translate all the formulae of *Precept II* and *Precept III* into modern notation.
2. Viète explains some of the formulae. Compare his explanations with those which you (or the textbook you use) give.
3. According to this passage, how does Viète regard the negative numbers.
4. The main mathematical preoccupation of the Greeks was geometric. How would they have demonstrated (or justified) the rules in *Precept III*.

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The Negative Numbers

Answer sheet: Viète

FRANÇOIS VIÈTE (VIETA)

1. Precept II

Viète	Equivalent in modern notation
Let there be two magnitudes A and B, and let the former be greater than the latter	A, B $A > B$
... and disjoined, they will be A "minus" B ...	$A - B$
... if "B 'plus' D" is to be subtracted from A, the remainder will be "A 'minus' B, 'minus' D" ...	$A -(B + D) = A - B - D$
But if D is already subtracted from B and "B 'minus' D" is to be subtracted from A, the result will be "A 'minus' B 'plus' D" . . .	$A -(B - D) = A - B + D$

Viète	Equivalent in modern notation
"A square = B plane" or "B plane = A square"	$\lvert A^2 - B \rvert$

Precept III

Viète	Equivalent in modern notation
"A in B"	AB, $A \cdot B$ or $A \times B$
If, however, the magnitudes to be multiplied, or one of them, be of two or more names....	$A(B + C)$ or $(A + B)(C + D)$
when the positive name of a magnitude is multiplied by a name also positive of another magnitude, the product will be positive,	$A(B + C)$ $= AB + AC$
and when it is multiplied by a negative name the product will be negative	$A(B - C)$ $= AB - AC$

(Note: from the test in Precept II, it is clear that, in this passage, Viète does not mean -B standing on its own, but rather B - C (or B = C), in which he will call, as later "-C" the "negative C".)

... as when "A = B" is multiplied by "D = G",	$(A - B)(D - G)$
since the product of the positive A and the negative G is negative ...	$\underline{A}(D - \underline{G})$ $= AD - \underline{AG}$
... the product of the negative B and the positive D is negative ...	$(A - \underline{B})\underline{D}$ $= AD - \underline{BD}$

(Note: By "in as much as A is, in accurately, brought forward as a magnitude to be multiplied", Viète means that we are required to multiply, not by A, but by A - B which is less than A. Hence in the product A(D - G) "too much is removed".)

Therefore, by way of compensation, when the negative B is multiplied by the negative G, the product is positive

$$(A \underline{- B})(D \underline{- G})$$

$$= AD - BD - AG \underline{+ BG}$$

(Note: Viète's justification, although based on arguments of magnitude, is essentially qualitative and not quantitative: we twice subtracted too much, so we have to add some on, but there is essentially no justification that the "compensation" is exact.)

2. The explanation will depend on the textbook and the level of the students. Thus A -(B + D) = A - B - D can be justified, for example, by movements on the number line, or by regarding the - in front of the brackets as equivalent to multiplication by -1, and then using the distributive law.

   The significance of this question is in the discussion of different approaches.

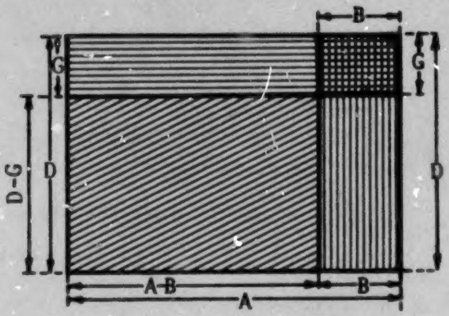
3. This point has been touched upon in the notes written in answer to Question 1, but we will detail it again here. If one reads the quotes from Viète superficially, one may get the impression that he deals with negative numbers and that he even "deduces" the law of signs in Precept III. But Viète did not admit negative numbers in any sense, as can be seen from

i) In order to perform the subtraction "A 'minus' B", the first quantity must be greater than the second.

ii) If one does not know which of the two quantities is the greater, then Viète uses the special symbol =, to indicate that whichever of the two is smaller is to be subtracted, i.e. to avoid obtaining a negative result, which he cannot admit.

iii) The law of signs is brought without his recognising negative numbers. Viète was not the first in this. The Greeks were aware of the extended distributive law (see also the answer to question 4) and that the product of two *subtracted numbers* (as opposed to *negative numbers*) gives a result which has to be added and not subtracted. Thus the use of law of signs does not imply acceptance of the negative numbers. In all our reading of historical source material, we have to be careful not to attach unjustified modern meaning to what we read. The strength of our modern symbolism is its generality, but in deducing that Viète's Precept III can be written in the form $(A - B)(C - D) = AC - AD - BC + BD$, we have to be careful to remember the limitations he imposes, i.e. $A > B$, $C > D$.

4. In answer to this question we do not expect a historical faithful demonstration (see later), but rather something like the following, which is also found in modern school texts.



The area of the rectangle whose sides are of length A - B and D - G is (A - B)(D - G). But, from the figure, we can obtain this area as the area of the large rectangle (A·D), less the area of the two bounding rectangular strips (A·G and B·D), except that we have removed the small rectangle (B·G) twice. So we have to add it back on once.

Since Viète was influenced by Diophantus*, it seems reasonable to suggest that he described in words and "rudimentary" symbolism, what the Greeks drew, so helping in the process of giving algebra an existence separate from geometry.

To get the flavour of an original Greek proof of what we today regard as an algebraic result, we bring Proposition 7 from Book II of *Euclid's Elements***.

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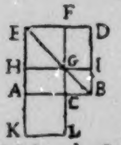
* Diophantus of Alexandria lived in the third century C.E. According to Kline, M., *Mathematical Thought from Ancient to Modern Times*, O.U.P., 1972, "The highest point of Alexandrian Greek algebra is reached with Diophantus." His major extent work is the apparently incomplete *Arithmetica*.

** Taken from the English version (7th edition) of de Chales' Euclid (London, 1726).

PROPOSITION VII.

A PROBLEM.

If a Line be divided, the Square of the whole Line with that of one of its Parts, is equal to two Rectangles contain'd under the whole Line, and that firſt Part, together with the Square of the other Part.



LET the Line AB be divided any where in C; the Square AD of the Line AB, with the Square AL, will be equal to two Rectangles contain'd under AB and AC, with the Square of CB. Make the Square of AB, and having drawn the Diagonal EB, and the Lines CF and HGI; prolong EA ſo far, as that AK may be equal to AC; ſo AL will be the Square of AC, and HK will be equal to AB; for HA is equal to GC,

and GC is equal to CB, becauſe CI is the Square of CB (by the Coroll. of the 4.)

Demonſtration.

'Tis evident, that the Squares AD and AL are equal to the Rectangles HL and HD, and the Square CI. Now the Rectangle HL is contain'd under HK equal to AB, and KL equal to AC. In like manner the Rectangle HD is contain'd under HI equal to AB, and HE equal to AC. Therefore the Squares of AB and AC are equal to two Rectangles contain'd under AB and AC, and the Square of CB.

In Numbers.

Suppoſe the Line AB to conſiſt of 9 Parts, AC of 4, and CB of 5. The Square of AB 9 is 81, and that of AC 4 is 16; which 81 and 16 added together make 97. Now one Rectangle under AB and AC, or four times 9, make 36, which taken twice, is 72; and the Square of CB 5 is 25; which 72 and 25 added together make alſo 97.

PRO

Algebraically, this result is equivalent to:

$$a^2 + b^2 = 2ab + (a - b)^2$$

where AB = a and AC = b.

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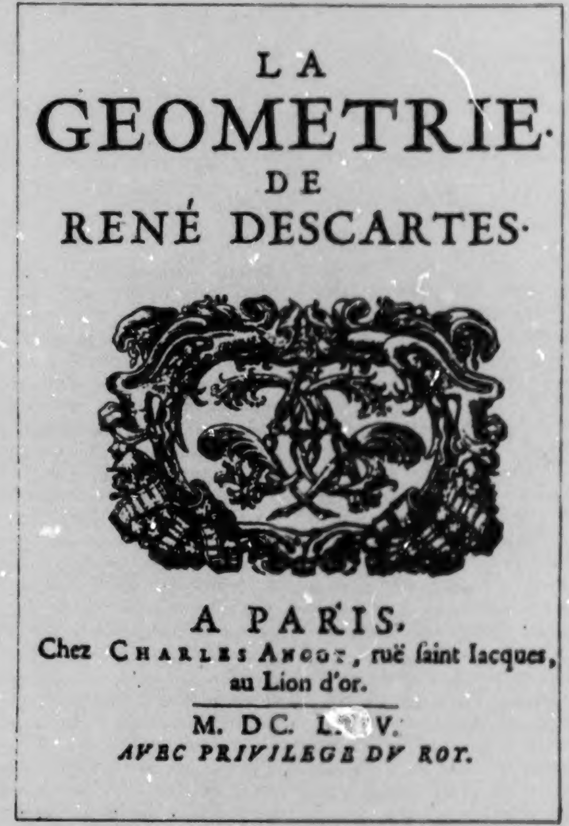
The Negative numbers

Worksheet: R. Descartes



LA
GEOMETRIE.
DE
RENÉ DESCARTES.

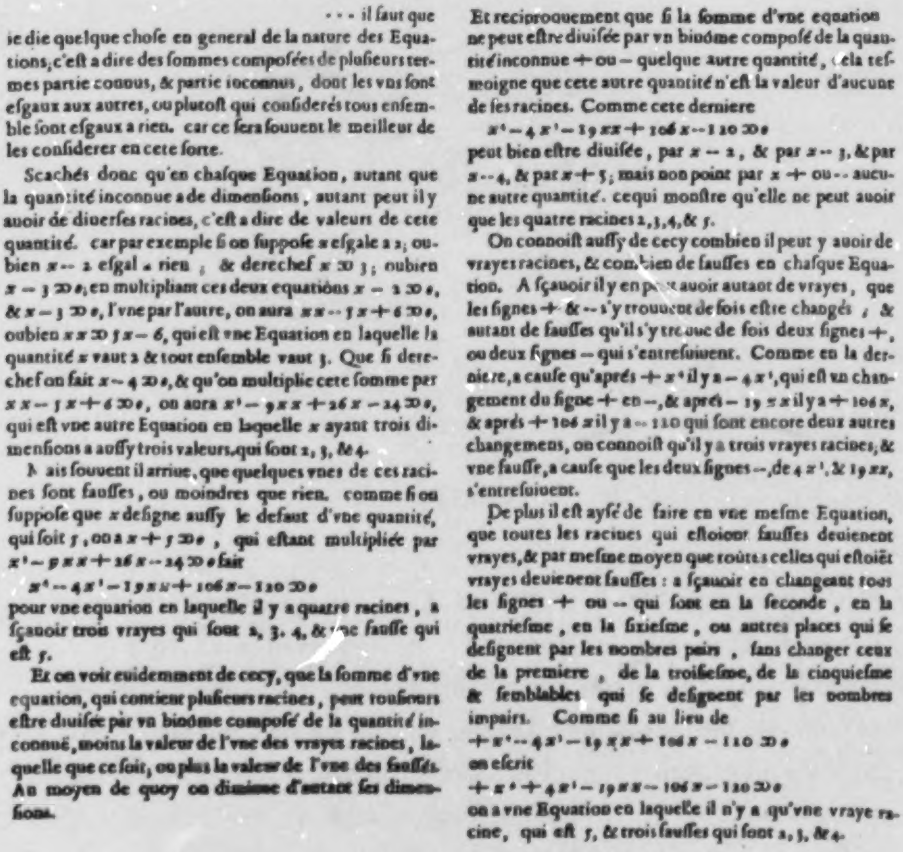
A PARIS.
Chez Charles Angot, ruë ſaint Iacques,
au Lion d'or.

M. DC. L[illegible]V.
AVEC PRIVILEGE DV ROY.

René Descartes (1596-1650) was a Frenchman, famous both as mathematician and philosopher. He received a religious (Christian) education and studied law. For some years he served in the army, and in the course of his duties, travelled to many European countries, where he met with mathematicians and philosophers. His most famous work is the *Discours de la Méthode* which was published in 1637. One of the appendices to this work is called *La Géometrie*. After some years this appendix began to be published as a separate work, as shown, for example, in the above title page of the 1664 edition.

Descartes is credited, among other things, with having made a further significant step in the process of developing analytical geometry, as well as the introduction of new mathematical symbols, some of which have remained in use to the present day.

The following is an extract from the first edition of *La Géometrie* as it appears in Smith D.E. & Latham M.L., *The Geometry of René Descartes*, Dover, 1954.



· · · il faut que ie die quelque choſe en general de la nature des Equations; c'eſt a dire des ſommes compoſées de pluſieurs termes partie connus, & partie inconnus, dont les vns ſont eſgaux aux autres, ou plutoſt qui conſiderés tous enſemble ſont eſgaux a rien. car ce ſera ſouuent le meilleur de les conſiderer en cete ſorte.

Sachés donc qu'en chaſque Equation, autant que la quantité inconnue a de dimenſions, autant peut il y auoir de diuerſes racines, c'eſt a dire de valeurs de cete quantité. car par exemple ſi on ſuppoſe x eſgale a 2; oubien $x - 2$ eſgal a rien; & derechef $x \propto 3$; oubien $x - 3 \propto 0$, en multipliant ces deux equations $x - 2 \propto 0$, & $x - 3 \propto 0$, l'vne par l'autre, on aura $xx - 5x + 6 \propto 0$, oubien $xx \propto 5x - 6$, qui eſt vne Equation en laquelle la quantité x vaut 2 & tout enſemble vaut 3. Que ſi derechef on fait $x - 4 \propto 0$, & qu'on multiplie cete ſomme par $xx - 5x + 6 \propto 0$, on aura $x^3 - 9xx + 26x - 24 \propto 0$, qui eſt vne autre Equation en laquelle x ayant trois dimenſions a auſſy trois valeurs, qui ſont 2, 3, & 4.

Mais ſouuent il arriue, que quelques vnes de ces racines ſont fauſſes, ou moindres que rien. comme ſi on ſuppoſe que x deſigne auſſy le defaut d'vne quantité, qui ſoit 5, on a $x + 5 \propto 0$, qui eſtant multipliée par $x^3 - 9xx + 26x - 24 \propto 0$ fait

$$x^4 - 4x^3 - 19xx + 106x - 120 \propto 0$$

pour vne equation en laquelle il y a quatre racines, a ſçauoir trois vrayes qui ſont 2, 3, 4, & vne fauſſe qui eſt 5.

Et on voit euidemment de cecy, que la ſomme d'vne equation, qui contient pluſieurs racines, peut touſiours eſtre diuiſée par vn binome compoſé de la quantité inconnuë, moins la valeur de l'vne des vrayes racines, laquelle que ce ſoit, ou plus la valeur de l'vne des fauſſes. Au moyen de quoy on diminue d'autant ſes dimenſions.

Et reciproquement que ſi la ſomme d'vne equation ne peut eſtre diuiſée par vn binome compoſé de la quantité inconnue + ou − quelque autre quantité, cela teſmoigne que cete autre quantité n'eſt la valeur d'aucune de ſes racines. Comme cete derniere

$$x^4 - 4x^3 - 19xx + 106x - 120 \propto 0$$

peut bien eſtre diuiſée, par $x - 2$, & par $x - 3$, & par $x - 4$, & par $x + 5$; mais non point par x + ou − aucune autre quantité. ce qui monſtre qu'elle ne peut auoir que les quatre racines 2, 3, 4, & 5.

On connoiſt auſſy de cecy combien il peut y auoir de vrayes racines, & combien de fauſſes en chaſque Equation. A ſçauoir il y en peut auoir autant de vrayes, que les ſignes + & − s'y trouuent de fois eſtre changés; & autant de fauſſes qu'il s'y trouue de fois deux ſignes +, ou deux ſignes − qui s'entreſuiuent. Comme en la derniere, a cauſe qu'aprés $+x^4$ il y a $-4x^3$, qui eſt vn changement du ſigne + en −, & aprés $-19xx$ il y a $+106x$, & aprés $+106x$ il y a -110 qui ſont encore deux autres changemens, on connoiſt qu'il y a trois vrayes racines; & vne fauſſe, a cauſe que les deux ſignes −, de $4x^3$, & $19xx$, s'entreſuiuent.

De plus il eſt ayſé de faire en vne meſme Equation, que toutes les racines qui eſtoient fauſſes deuienent vrayes, & par meſme moyen que toutes celles qui eſtoiēt vrayes deuienent fauſſes: a ſçauoir en changeant tous les ſignes + ou − qui ſont en la ſeconde, en la quatrieſme, en la ſixieſme, ou autres places qui ſe deſignent par les nombres pairs, ſans changer ceux de la premiere, de la troiſieſme, de la cinquieſme & ſemblables qui ſe deſignent par les nombres impairs. Comme ſi au lieu de

$$+x^4 - 4x^3 - 19xx + 106x - 110 \propto 0$$

on eſcrit

$$+x^4 + 4x^3 - 19xx - 106x - 120 \propto 0$$

on a vne Equation en laquelle il n'y a qu'vne vraye racine, qui eſt 5, & trois fauſſes qui ſont 2, 3, & 4.

Questions

1. Complete the "dictionary" in the following table.

Descartes	Today
quantité inconnue	
	degree of an equation
--	
rien	
	=
	x^2
	additive inverse (of a positive number)
racines fausses	
somme d'une équation	

2. What does Descartes have to say about the connection between the number of roots and the degree of an equation?

3. Using the method which Descartes suggests, construct the equation with the two "true" roots 1 and 2, and the two "false" roots 2 and 3.

4. Descartes states a necessary and sufficient condition for a polynomial $P(x)$ to be divisible by the expression $x - a$.
What is this condition? Prove it!

5. In the equation given in the text Descartes makes an assertion about the number of its positive and negative roots.
   a. Which fact does he use in his argument?
   b. Is his conclusion justified?
      (Consider for example $x^3 + x^2 + x - 3 = 0$)

6. Descartes discusses the connection between the number of positive and negative roots and the number of changes or non-changes of sign in the terms of an equation. The resulting rule has become known as *Descartes' Rule of Signs*.
   a) How would you formulate the rule in the light of your answer to Qu. 5?
   b) How would you justify the rule in general?

7. a) What change does Descartes suggest in a given equation in order to turn positive into negative roots and vice-versa?
   b) Prove that it works.

8. Compare Descartes' and Viète's attitude to negative numbers.

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The Negative Numbers

Answer sheet: R. Descartes

Note →
the mistake
in the title
of the book.

1.

Descartes	Today
quantité inconnue	unknown, variable
dimensions de la quantité inconnue	degree of an equation
--	-
rien	zero
ꝏ	=
xx	x^2
le défaut d'une quantité	additive inverse (of a positive number)
racine fausse	negative root
somme d'une équation	polynomial (or function)*

* Descartes says that an equation can be written in two forms; for example,

xx ꝏ 5x -- 6

or xx -- 5x + 6 ꝏ 0 ,

which we would write as

$$x^2 = 5x - 6$$

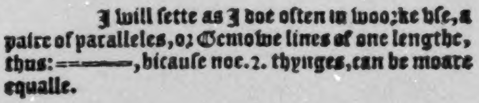
or $$x^2 - 5x + 6 = 0$$

Following some of his predecessors, Descartes prefers the latter form. The left-hand side of the equation he calls *somme d'une équation*, which we would call a polynomial (or function).

★★★

As mentioned in the worksheet, Descartes has an important place in the development of mathematical symbolism, and we will enlarge a little on one or two aspects*.

- The symbol ꝏ to designate equality. Some explain this symbol as a combination of the letters a and e (æ) and their reverse. These two letters are the first letters of the Latin *aequalis*, meaning "equal". Others maintain that ꝏ is astronomic symbol for *Taurus*, and yet others that it is an arbitrary invention. The = symbol which we use today, apparently first appeared in the book *Whetstone of Witte*, published in London in 1557 by Robert Recorde (1510 - 1558), who wrote



I will ſette as I doe often in woorke vſe, a
paire of paralleles, or Gemowe lines of one lengthe,
thus: =====, bicauſe noe .2. thynges, can be moare
equalle.

  That is, no two things are "more equal" than two parallel lines. The fact that Descartes, nearly a century later, was using ꝏ , shows that it took a long while for the = symbol to become universally established.

- The use of small letters in algebra. This usage remained with us. In particular, the use of the last letters of the alphabet, x, y, z to represent unknowns, and the first letters, a, b, c to denote known quantities, is due to Descartes.

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[*] Further details on Descartes' symbolism, and on the history of mathematical symbolism in general, can be found in Cajori, F., *A History of Mathematical Notation*, Vol. I, Open Court, 1928.

- The designation of exponents by a small number to the upper right of the letter as in x^3 is also due to Descartes, although he himself occasionally writes (as in the extract) xx instead of x^2.
- The use of -- for subtraction. This symbol is characteristic of a number of works in the first half of the seventeenth century.

2. Descartes writes:

> Scachés donc qu'en chaſque Equation, autant que la quantité inconnue a de dimenſions, autant peut il y auoir de diuerſes racines, c'eſt a dire de valeurs de cete quantité.

Today we have the *Fundamental Theorem of Algebra* which states that an equation of degree n has n roots, where all roots are taken into account - positive, negative, equal (according to their multiplicity) and complex. Descartes writes *peut il y avoir* and not "must have" or "has", since, apparently, he was referring to real roots and then only to different roots - a multiple root probably being counted only once. Hence the number of roots is equal to or less than the degree of the equation (less, if there are complex or multiple roots).

The fundamental theorem of algebra was, it seems, first enunciated by Girard (1595-1632). Partial attempts at a proof were made by Euler (1707-1783) and D'Alembert (1717-1783). The first complete was given by Gauss (1777-1855) in his doctoral thesis in 1799*.

The proof of the theorem is not easy. But, if we assume that every polynomial equation has at least one root (real or complex), then the remainder becomes easy to prove. For if

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* Further details and sources connected with this theorem can be found in, for example, Struik, D.J., *A Source-Book in Mathematics, 1200-1800,* Harvard U.P., 1969.

a is the root, divide the polynomial by $x - a$, to obtain a new polynomial of degree one less than the original. This new polynomial also has at least one root, by our assumption, and hence the process can be continued until the original polynomial has been completely factorised. Clearly the number of roots is equal to the degree of the equation.

3. $(x - 1)(x - 2) = x^2 - 3x + 2$ and hence $x^2 - 3x + 2 = 0$ is an equation with the two "true" (positive) roots 1 and 2. If we now multiply by $x + 2$ and then by $x + 3$, we shall obtain an equation with four roots: the two "true" roots 1 and 2 and the two "false" (negative) roots 2 and 3.

$$(x^2 - 3x + 2)(x + 2) = x^3 - x^2 - 4x + 4$$

$$(x^3 - x^2 - 4x + 4)(x + 3) = x^4 + 2x^3 - 7x^2 - 8x + 12.$$

4. Descartes maintained that the necessary and sufficient condition for a polynomial $P(x)$ to be divisible by $x - a$, in that a be a root of the equation $P(x)$, i.e. $P(a) = 0$.

Given the polynomial $P(x)$, we have to prove that

$P(x)$ is divisible by $(x - a)$ $\Leftrightarrow$ $P(a) = 0$, that is
a root of $P(x) = 0$.

i) We start by proving $\Rightarrow$.

Suppose $P(x)$ is divisible by $x - a$, that is

$$P(x) = (x - a)\, Q(x),$$

where $Q(x)$ is a polynomial of degree one less than $P(x)$. If we substitute x for a, we obtain

$$P(a) = (a - a)\, Q(a) = 0$$

ii) Now to prove $\Leftarrow$.

Suppose that $P(a) = 0$ and that, contrary to what we have to prove, $P(x)$ is not divisible by $x - a$, that is

$$P(x) = (x - a)\,Q(x) + R,$$

where $R \neq 0$. Substituting x for a, we obtain

$$P(a) = 0 + R \neq 0.$$

But we begin by supposing $P(a) = 0$. Hence the supposition that $P(x)$ is not divisible by $x - a$ leads us to a contradiction. Thus we conclude that $P(a) = 0 \Rightarrow P(x)$ divisible by $(x - a)$.

5. a. Descartes states with care the following rule:

> . . . y en peut auoir autant de vrayes, que les ſignes + & -- s'y trouuent de fois eſtre changés ; & autant de fauſſes qu'il s'y trouue de fois deux ſignes +, ou deux ſignes -- qui s'entreſuiuent.

Afterwards, he gives an example and concludes from it

> . . . on connoiſt qu'il y a . . .

b. Consider the example

$$x^3 + x^2 + x - 3 = 0$$

and following Descartes we conclude that the equation has 2 negative roots (two non-changes of sign) and one positive root (one change of sign). But this is not correct: the equation has one positive root 1, and the other two roots are not real.

6. a) Descartes' conclusion is justified on condition that all the roots are real. Otherwise, the rule has to be stated as follows:

The number of positive roots is less than or equal to the number of changes of sign in successive terms of the polynomial, and the number of negative roots is less than or equal to the number of non-changes of sign. As already mentioned, Descartes referred to real numbers and different roots only, and hence he wrote *peut-il y avoir.*

b) Consider the signs in some polynomial expression, for example

+ + - - + - - - + - + -

If we multiply this polynomial by the binomial + - (that is the binomial x - a, a positive), we obtain

```
  + + - - + - - - + - + -      (seven changes of sign)
x
  + -
  -----------------------
  + + - - + - - - + - + -
    - - + + - + + + - + - +
  -------------------------
  + ± - ± + - ± ± + - + - +
    ?   ?     ? ?
```

Now consider the "worst" possible case, that is the one in which we obtain the least number of changes of sign, and we still must have at least 8, that is at least one more than when we started.

We can state that we have done as follows: we multiplied by x - a (a positive), that is we have constructed a new equation, which has the additional positive root a, and as a consequence we have increased the number of changes of sign by at least one. Hence, a polynomial equation cannot have more positive roots than changes of sign.

A similar argument gives the result for negative roots and non-changes of sign.

7. a) The required change is to alter the sign of every term in an even position in the equation. For example,

$$x^3 + 6x^2 + 11x + 6 = 0$$

has three negative roots, -1, -2, -3 . If we make the required change we have

$$x^3 - 6x^2 + 11x - 6 = 0$$

which has the three positive roots 1, 2 and 3 . The rule is correct on condition that all the powers of x appear in the equation - even if with zero coefficient.

b) Without loss of generality, we may assume that the even positions are occupied by odd powers of x. (For example, the equation in part (a) can be rewritten by adding $0x^4$ at the beginning.) Then changing their sign is equivalent to replacing

$$f(x) = 0$$

by $$f(-x) = 0$$

and the roots of the latter are of opposite sign to the roots of the former.

8. Although Descartes does not give full recognition to the negative numbers (he refers to false roots and to -5 as the "defect" of the quantity 5), nevertheless there is a distinct advance on the position of Viète. Descartes does refer to negative numbers and deal with them as roots of an equation.

RENE DESCARTES

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The Negative numbers

Worksheet: Contradictions in the use of negative numbers

Part A: Antoine Arnauld

Antoine Arnauld (1612-1694) was a Frenchman, theologian and mathematician. He was close to the mathematician and philosopher B. Pascal (1623-1662). One of Arnauld's best known works is the *Nouveaux Eléments de Géometrie* (1667)*.

The following is a quotation** from a letter by Arnauld addressed to Prestet (1648-1691), in which the central issue is Arnauld's reservations with regard to negative numbers.

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* More details about Arnauld and his work can be found in: Coolidge J.L., *The Mathematics of Great Amateurs*, Oxford,1949.

** As it is brought in: Schrecker P., Arnauld, Malebranche, Prestet et la Theorie des Nombres Negatifs, *Thales* 1935,vol.2,82-90.

.... je ne comprens pas que le quarré de — 5 puisse être la même chose que le quarré de + 5, et que l'un et l'autre soit + 25. Je ne sçai de plus comment ajuster cela au fondement de la multiplication, qui est que l'unité doit être à l'une des grandeurs que l'on multiplie, comme l'autre est au produit. Ce qui est également vrai dans les entiers et dans les fractions. Car 1 est à 3, comme 4 est à 12. Et 1 est à 1/3, comme 1/4 est à 1/12. Mais je ne puis ajuster cela aux multiplications de deux moins. Car dira-t-on que + 1 est à — 4, comme — 5 est à + 20 ? Je ne le vois pas. Car + 1 est plus que — 4. Et au contraire — 5 est moins que + 20. Au lieu que dans toutes les autres proportions, si le premier terme est plus grand que le second, le troisiéme doit être plus grand que le quatriéme.

Questions

1. Explain using modern notation what Arnauld means by the *fondement de la multiplication.*

2. Which properties of negative numbers does Arnauld use in his argument?

3. What is the "contradiction" that he sees in the use of negative numbers?

4. How would you answer him?

Part B: John Wallis

IOHANNES WALLIS D·D

GEOMETRIAE PROF· SAVILIANVS OXONIAE

REGALIS SOCIETATIS LONDINI SODALIS.

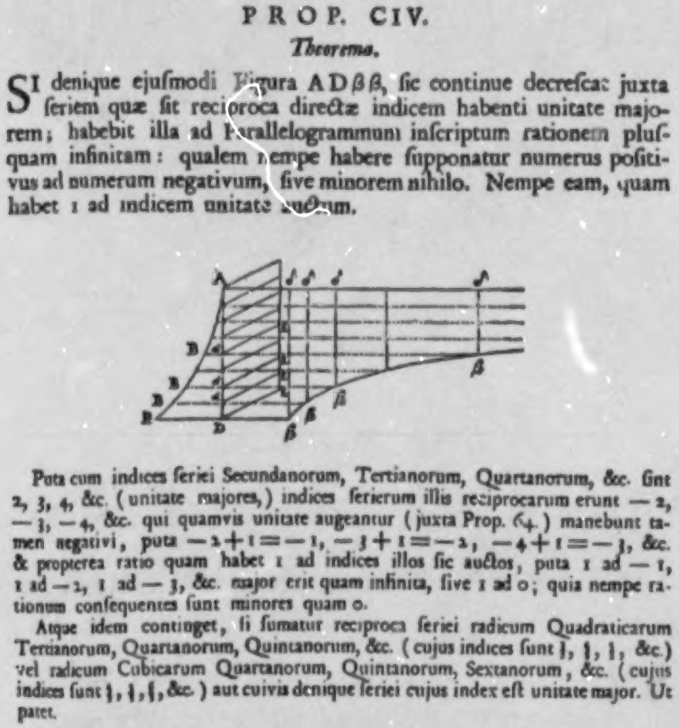
John Wallis (1616-1703) was an Englishman, Savilian Professor of Geometry in Oxford, and a founder member of the Royal Society.

Scott (1981)* writes about Wallis:

... more than any other the precursor of the mighty Newton. ... it seems beyond all doubt that amongst his contemporaries at least Wallis was esteemed eminent, ... over a range of subjects which was truly encyclopaedic. Mathematics, mechanics, sound, philology, the phenomena of the tides, even music - in all these his writings give evidence of profound knowledge ...

Among Wallis' works are: *Arithmetica Infinitorum* (1655) and his "textbook": *A Treatise on Algebra both Historical and Practical* (1685).

The following is a passage from the *Arithmetica Infinitorum*, page 409:



PROP. CIV.

Theorema.

SI denique ejuſmodi Figura ADββ, ſic continue decreſcat juxta ſeriem quæ ſit reciproca directæ indicem habenti unitate majorem; habebit illa ad Parallelogrammum inſcriptum rationem pluſquam infinitam: qualem nempe habere ſupponatur numerus poſitivus ad numerum negativum, ſive minorem nihilo. Nempe eam, quam habet 1 ad indicem unitate auctum.

Puta cum indices ſeriei Secundanorum, Tertianorum, Quartanorum, &c. ſint 2, 3, 4, &c. (unitate majores,) indices ſerierum illis reciprocarum erunt −2, −3, −4, &c. qui quamvis unitate augeantur (juxta Prop. 64.) manebunt tamen negativi, puta $-2+1=-1$, $-3+1=-2$, $-4+1=-3$, &c. & propterea ratio quam habet 1 ad indices illos ſic auctos, puta 1 ad −1, 1 ad −2, 1 ad −3, &c. major erit quam infinita, ſive 1 ad 0; quia nempe rationum conſequentes ſunt minores quam 0.

Atque idem continget, ſi ſumatur reciproca ſeriei radicum Quadraticarum Tertianorum, Quartanorum, Quintanorum, &c. (cujus indices ſunt $\frac{3}{2}$, $\frac{4}{2}$, $\frac{5}{2}$, &c.) vel radicum Cubicarum Quartanorum, Quintanorum, Sextanorum, &c. (cujus indices ſunt $\frac{4}{3}$, $\frac{5}{3}$, $\frac{6}{3}$, &c.) aut cuivis denique ſeriei cujus index eſt unitate major. Ut patet.

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* Scott J.F., *The Mathematical Work of John Wallis.* Chelsea Pub. Co., N.Y., 1981.

In this passage Wallis mentions:

... rationem plusquam infinitam: qualem nempe habere supponatur numerus positivus ad numerum negativum, five minorem nihilo.

This means: "a ratio greater than infinity, such as a positive number may be supposed to have to a negative number, or one less than nothing"*.

This conclusion (in modern terminology) is "deduced" from the sequence:

$$\ldots \quad \frac{1}{5} \quad \frac{1}{4} \quad \frac{1}{3} \quad \frac{1}{2} \quad \frac{1}{1} \quad \frac{1}{0} \quad \frac{1}{-1} \quad \frac{1}{-2} \quad \frac{1}{-3} \quad \ldots$$

in which the ratios increase from the first to the sixth, i.e. from $\frac{1}{5}$ to $\frac{1}{0}$, which "was shown" to be infinite*.

So if one continues to the right, one gets:

$$\frac{1}{0} < \frac{1}{-1} < \frac{1}{-2} \quad \ldots$$

and this means that the ratio of a positive to a negative is greater than infinity.

Questions

5. Write the general term of the sequence and draw its graph (as function of n).

6. How would you answer Wallis' argument?

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* See Scott ibid. p. 44-45.

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The Negative numbers

Answer sheet: Contradictions in the use of negative numbers

Part A: Antoine Arnauld

1. Arnauld says that a basic principle of multiplication (of two factors) is that the ratio of unity to one factor is equal to the ratio of the second factor to the product; i.e., in the product ab

$$\frac{1}{a} = \frac{b}{ab} \quad \text{or} \quad \frac{1}{b} = \frac{a}{ab}$$

2. In his argument, Arnauld uses:

   i) the assumption of order as for example in

   $$1 > -4 \, , \quad -5 < 20$$

   ii) (implicitly) the law of signs in multiplication, and specifically a negative times a negative (or alternatively the law of signs in division, if we regard the ratios 1 to -4 and -5 to 20 as such).

3. Arnauld states that if one accepts that the multiplication of two minuses is a plus, for instance

$$(-4)\cdot(-5) = +20 \, ,$$

then the "basic principle of multiplication" (see Qu. 1) implies that

$$\frac{1}{-4} = \frac{-5}{20}$$

that is to say that, a bigger number is to a smaller (1 to -4) as a smaller number is to a bigger (-5 to 20) and this contradicts the proportion concept:

Si le premier terme est plus grand que le second,
le troisième doit être plus grand que le quatrième.

4. Before commenting Arnauld's "contradiction" in modern terms, we bring two answers as given by Arnauld's contemporaries.

   i) J. Prestet * answered to Arnauld's letter, in which he says:

> Il est clair que le plus et le moins des grandeurs égales se font l'un à l'autre des retranchements réciproques, aprés lesquels il ne reste rien, c'est-à-dire que comme le plus d'une grandeur en détruit et retranche le moins, le moins pareillement en détruit et retranche le plus. Que + 1 par exemple retranche — 1, comme — 1 retranche + 1 ; ou ce qui signifie la même chose, que + 1 — 1 = 0, comme — 1 + 1 = 0. Il est donc égal de dire le plus d'une grandeur, ou le retranchement de son moins. Or le retranchement du moins signifie encore la même chose que moins moins. Il sera donc égal de dire plus, ou de dire moins moins. Et ainsi ce sera la même chose de dire + 5 fois 5, ou + 25 ; ou de dire — — 5 fois 5, ou — — 25. Or — — 25 est le quarré de — 5. Car multiplier — 5 par — 5, c'est prendre — 5 autant de fois qu'il y a d'unitez dans — 5, c'est à dire — 5 fois : ou ce qui est égal, c'est ôter — 5 autant de fois qu'il y a d'unitez dans 5 ; je ne dis pas — 5, parce que ce moins est transplacé dans le mot d'*ôter* qui le signifie. Or je ne puis ôter une fois — 5, qu'en mettant une fois + 5. Et ainsi je ne puis ôter 4 fois — 5, qu'en mettant 4 fois + 5 ou + 20. Cela paroit démonstratif : et c'est de quoi je voulois établir le principe dans ma lettre, lorsque j'y marquois que si un homme qui n'a rien et qui ne doit rien, recevait la diminution ou le retranchement d'une dette supposée de 1000 écus, ou que

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* According to Schreker, ibid.

si on luy ôtoit — 1000 écus, il auroit ensuite + 1000 écus, parce qu'on n'auroit pû luy ôter cette dette qu'en luy donnant + 1000 écus de quoy la payer. Et comme on suppose qu'il ne doit rien, il seroit en droit de garder ce qu'il auroit receu. De sorte qu'il auroit véritablement 1000 écus.

« Si donc + 1 détruit — 1, comme — 1 détruit + 1 ; ou si + 1 est une fois ôté ou retranché dans — 1, comme — 1 une fois ôté ou retranché dans + 1 : ces termes *une fois ôté ou retranché*, et l'expression plus abrégée — 1, exposeront également ce que + 1 est à — 1, et ce que — 1 est à + 1. Il est assez évident par les *Nouveaux Elémens de Géométrie* que plus en moins, ou moins en plus, donne également moins. Ils semblent même supposer, comme une vérité constante, qu'on aura prouvé que moins en plus donne moins, lorsqu'on aura prouvé que plus en moins donne moins. Il sera donc constant par vos propres principes, que + 1 divisé par — 1, et que — 1 divisé par + 1, donneront un même exposant — 1 ; ou que + 1 est au juste contenu dans — 1, comme — 1 dans + 1. Or il est clair que partout où les exposants sont les mêmes, les rapports sont égaux ; les différences ne sont pas l'essentiel des proportions géométriques, elles le sont seulement des arithmétiques. Il suffit pour la géométrique que le premier terme soit dans le second de la même sorte que le troisiéme est dans le quatriéme. Cette notion est si étendue qu'elle embrasse les entiers, les fractions, les incommensurables, et les grandeurs négatives et positives. Il est donc permis de raisonner ainsi, $+1 : -1 = -1 : +1$. Or $+1 : -1 = +5 : -5 = +4 : -4$. Et $-1 : +1 = -5 : +5 = -4 : +4$. Donc $+1 : -1 = -5 : +5 = -4 : +4$. Et par un changement alterne, $+1 : -5 = -1 : +5$. Et $+1 : -4 = -1 : +4$. Or $-1 : +5 = -5 : +25$. Et $-1 : +4 = -5 : +20$. Donc par égalité $+1 : -5 = -5 : +25$. Et $+1 : -4 = -5 : +20$. Et s'il est vrai que l'unité est ôtée ou retranchée 5 fois dans — 5, et 4 fois dans — 4, comme — 5 est ôté ou retranché 5 fois dans + 25, et 4 fois dans + 20 ; on trouvera encore en divisant + 25 par — 5, et — 5 par + 1, un même exposant — 5 ; et en divisant + 20 par — 5, et — 4 par + 1, un même exposant — 4. De sorte qu'il est clair en toutes manières, que moins multiplié par moins donne plus, essentiellement et par luy-même.

ii) G.W. Leibniz (1646-1716), the German mathematician who was the "co-founder" of the calculus wrote an answer to Arnauld's argument in 1712*, in which he agreed that these are not truly ratios [*Veras illas ratione non esse*]. But he argued that one can calculate with these ratios as well as one does with imaginary quantities, so they are tolerable [*toleranter verae*] and useful,

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* Leibniz G.W., *Math. Schriften*, Vol. 5, Ch. 29, p.387-389.

[*habent tamen usum magnum in calculando*] in spite of their lack of rigor [*Rigorem quidem non sustinent*].

The following is the full quotation:

XXIX.

OBSERVATIO QUOD RATIONES SIVE PROPORTIONES NON HABEANT LOCUM CIRCA QUANTITATES NIHILO MINORES, ET DE VERO SENSU METHODI INFINITESIMALIS. *)

Cum olim Parisiis Vir summus *Antonius Arnaldus* sua nova Geometriae Elementa mecum communicaret, atque in iisdem admirari se testatus fuisset, quomodo posset esse 1 ad -1, ut -1 ad 1, quae res probari videtur ex eo, quod productum est idem sub extremis quod sub mediis, cum utroque prodeat $+1$; jam tum dixi mihi videri, *veras* illas *rationes* non esse, in quibus quantitas nihilo minor est antecedens, vel consequens, etsi in calculo haec, ut alia *imaginaria*, tuto et utiliter adhibeatur. Et sane identitatis rationum verarum fundamentum est rerum similitudo, quae facit exempli causa, ut segmentis similibus diversorum circulorum assumtis sit ubique eadem ratio chordae ad radium, seu ut chorda minoris se habeat ad radium minoris, vel ut chorda majoris ad radium majoris. Sed vero nulla plane apparet similitudo in supradicta Analogia. Si enim -1 est minus nihilo, utique 1 ad -1 erit ratio majoris ad minus; sed vero contra ratio -1 ad 1 est ratio minoris ad majus; quomodo ergo utrobique eadem ratio erit? Sed rationes istas esse imaginarias, etiam alio certissimo argumento comprobabo, scilicet a Logarithmis. Nempe ratio, cui nullus datur respondens Logarithmus, ratio vera non est. Porro posito unitatis Logarithmum esse 0, rationis -1 ad 1 idem est Logarithmus, qui ipsius -1; at ipsius -1 non datur Logarithmus. Non enim est positivus, nam talis omnis est Logarithmus numeri positivi unitate majoris. Sed tamen etiam non est negativus, quia talis omnis est Logarithmus numeri positivi unitate minoris. Ergo Logarithmus ipsius -1 cum nec positivus sit nec negativus, superest ut sit non verus, sed imaginarius. Itaque et ratio, cui respondet, ***non vera***, sed ***imaginaria*** erit. Idem etiam sic probo: Si daretur verus Logarithmus ipsius -1, seu rationis -1 ad 1, ejus logarithmi dimidium foret Logarithmus ipsius $\sqrt{-1}$, sed $\sqrt{-1}$ est quantitas imaginaria. Itaque daretur Logarithmus verus imaginariae quantitatis, quod est absurdum. Et proinde nonnihil humani passus est insignis in paucis Geometra ***Johannes Wallisius***, cum dixisset rationem 1 ad -1 esse plus quam infinitam; et recte hoc (etsi aliis considerationibus) celeberrimus ***Varignonius*** rejicit. Interim nolim cum ipso negare, -1 esse quantitatem

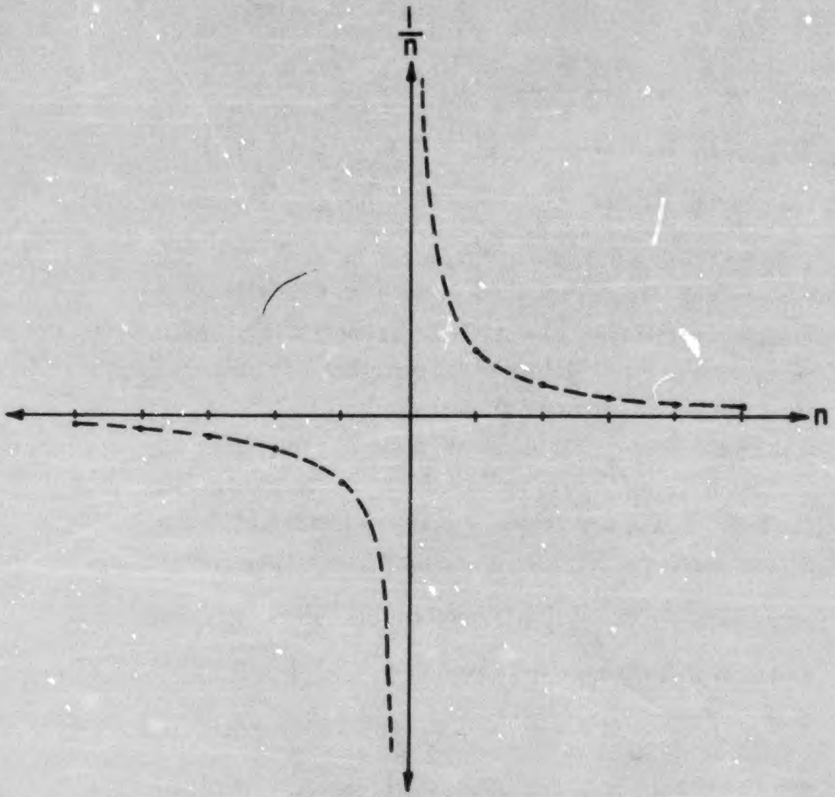
nihilo minorem, modo id sano sensu intelligatur. Tales enuntiationes sunt *toleranter verae*, ut ego cum summo Viro *Joachimo Jungio* loqui soleo; Galli appellarent *passables*. Rigorem quidem non sustinent, habent tamen usum magnum in calculando, et ad artem inveniendi, universalesque conceptus valent. Talis fuit locutio Euclidis, cum Angulum contactus dixit esse rectilineo quovis minorem; tales sunt multae Geometrarum aliae, in quibus est figuratum quodammodo et crypticum dicendi genus. Sunt tamen quidem, ut sic dicam, *tolerabilitatis*. Porro, ut nego rationem, cujus terminus sit quantitas nihilo minor, esse realem, ita etiam nego, proprie dari numerum infinitum vel infinite parvum, etsi Euclides saepe, sed sano sensu, de linea infinita loquatur. *Infinitum continuum* vel *discretum* proprie nec unum, nec totum, nec quantum est, et si analogia quaedam pro tali a nobis adhibeatur, ut verbo dicam, est modus loquendi; cum scilicet plura adsunt, quam ullo numero comprehendi possunt, numerum tamen illis rebus attribuemus analogice, quem infinitum appellamus. Itaque jam olim judicavi, cum infinite parvum esse errorem dicimus, intelligi dato quovis minorem, revera nullum; et cum ordinarium, et infinitum, et infinities infinitum conferimus, perinde esse ac si conferremus ascendendo diametrum pulvisculi, diametrum terrae, et diametrum orbis fixarum, aut his quantumvis (per gradus) majora minoraque, eodemque sensu descendendo diametrum orbis fixarum, diametrum terrae, et diametrum pulvisculi posse comparari ordinario, infinite parvo, et infinities infinite parvo, sed ita ut quodvis horum in suo genere quantumvis majus aut minus concipi posse intelligatur. Cum vero saltu ad ultimum facto ipsum infinitum aut infinite parvum dicimus, commoditati expressionis seu breviloquio mentali inservimus, sed non nisi *toleranter vera* loquimur, quae explicatione *rigidantur*. Atque haec etiam mea sententia est de areis illis Hyperboliformium Asymptoticis, quae infinitae, infinitiesque infinitae esse dicuntur, id est talia rigorose loquendo vera non esse posse, tamen sano aliquo sensu tolerari. Atque haec tum ad terminandas virorum clarissimorum Varignonii et Grandii controversias, tum ad praecavendos chimericos quosdam conceptus, tum denique ad elidendas oppositiones contra methodum *infinite similem* prodesse possunt.

In modern terms we can answer Arnauld as follows. The ratio concept and the concept of "larger" and "smaller" (i.e. order) to Arnauld are those to which he was used in the set of positive numbers. Now he extends these concepts to a larger set (i.e. includes the negative numbers), and there is no a priori reason why these properties can be carried over without creating contradictions.

In our case the relationship between order and Proportion ~~proposition~~ is a property of positive numbers only: we cannot have them both in the enlarged set. Hence if we chose to maintain the order concept, in the form to which we are used, the idea of ratio of smaller to larger must go. Another example is the following: in the set of positive numbers $a < 1 \Rightarrow ab < b$. This property does not carry over to the set of all numbers. In a subsequent worksheet (Peacock) we shall deal with this point at length.

Part B: John Wallis

5. The general term of the sequence in Wallis' argument is $\frac{1}{n}$, and its graph is:



6. The great Swiss mathematician Leonard Euler (1707-1783) was one of the most prolific of all time*. In his book *Differenzial Rechnung* Euler answered Wallis argument. The following is a quotation from the 1790 German edition.

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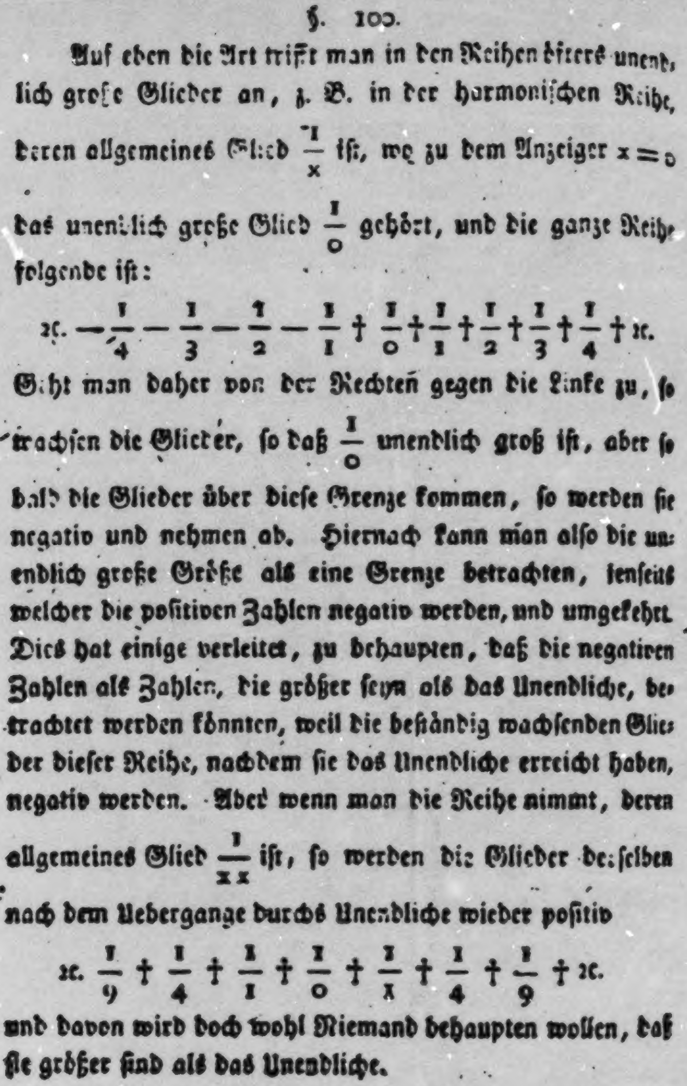
* More details about Euler can be found in the Euler worksheet in this series.



§. 100.

Auf eben die Art trifft man in den Reihen öfters unendlich groſſe Glieder an, z. B. in der harmoniſchen Reihe, deren allgemeines Glied $\frac{1}{x}$ iſt, wo zu dem Anzeiger $x = 0$ das unendlich große Glied $\frac{1}{0}$ gehört, und die ganze Reihe folgende iſt:

$$\text{ꝛc.} - \frac{1}{4} - \frac{1}{3} - \frac{1}{2} - \frac{1}{1} + \frac{1}{0} + \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \text{ꝛc.}$$

Sieht man daher von der Rechten gegen die Linke zu, ſo wachſen die Glieder, ſo daß $\frac{1}{0}$ unendlich groß iſt, aber ſo bald die Glieder über dieſe Grenze kommen, ſo werden ſie negativ und nehmen ab. Hiernach kann man alſo die unendlich große Größe als eine Grenze betrachten, jenſeits welcher die poſitiven Zahlen negativ werden, und umgekehrt. Dies hat einige verleitet, zu behaupten, daß die negativen Zahlen als Zahlen, die größer ſeyn als das Unendliche, betrachtet werden könnten, weil die beſtändig wachſenden Glieder dieſer Reihe, nachdem ſie das Unendliche erreicht haben, negativ werden. Aber wenn man die Reihe nimmt, deren allgemeines Glied $\frac{1}{xx}$ iſt, ſo werden die Glieder derſelben nach dem Uebergange durchs Unendliche wieder poſitiv

$$\text{ꝛc.}\ \frac{1}{9} + \frac{1}{4} + \frac{1}{1} + \frac{1}{0} + \frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \text{ꝛc.}$$

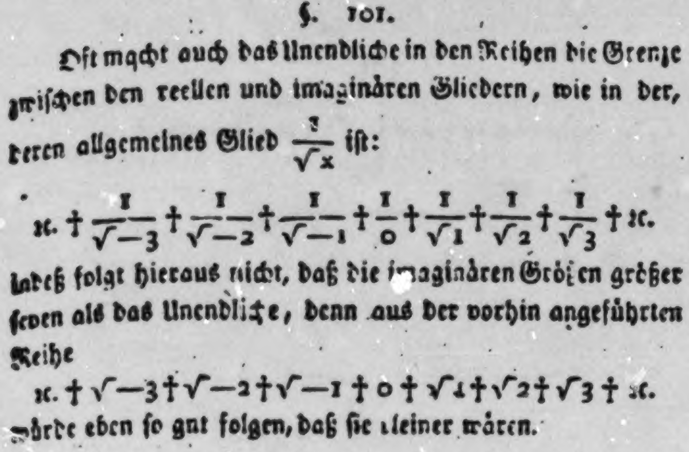
und davon wird doch wohl Niemand behaupten wollen, daß ſie größer ſind als das Unendliche.

In summary, he states that from sequences of reciprocals one cannot conclude anything. He exemplifies his rejection of the Wallis argument with $\frac{1}{n^2}$,

$$\ldots, \frac{1}{9}, \frac{1}{4}, \frac{1}{1}, \frac{1}{0}, \frac{1}{1}, \frac{1}{4}, \frac{1}{9}, \ldots$$

from which "no one will assert that they (the positive terms on the right) are greater than infinity".

Euler rubs in the point that this form of argument only leads to contradictions, by giving another example



§. 101.

Oft macht auch das Unendliche in den Reihen die Grenze zwiſchen den reellen und imaginären Gliedern, wie in der, deren allgemeines Glied $\frac{1}{\sqrt{x}}$ iſt:

$$\text{ꝛc.} + \frac{1}{\sqrt{-3}} + \frac{1}{\sqrt{-2}} + \frac{1}{\sqrt{-1}} + \frac{1}{0} + \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \text{ꝛc.}$$

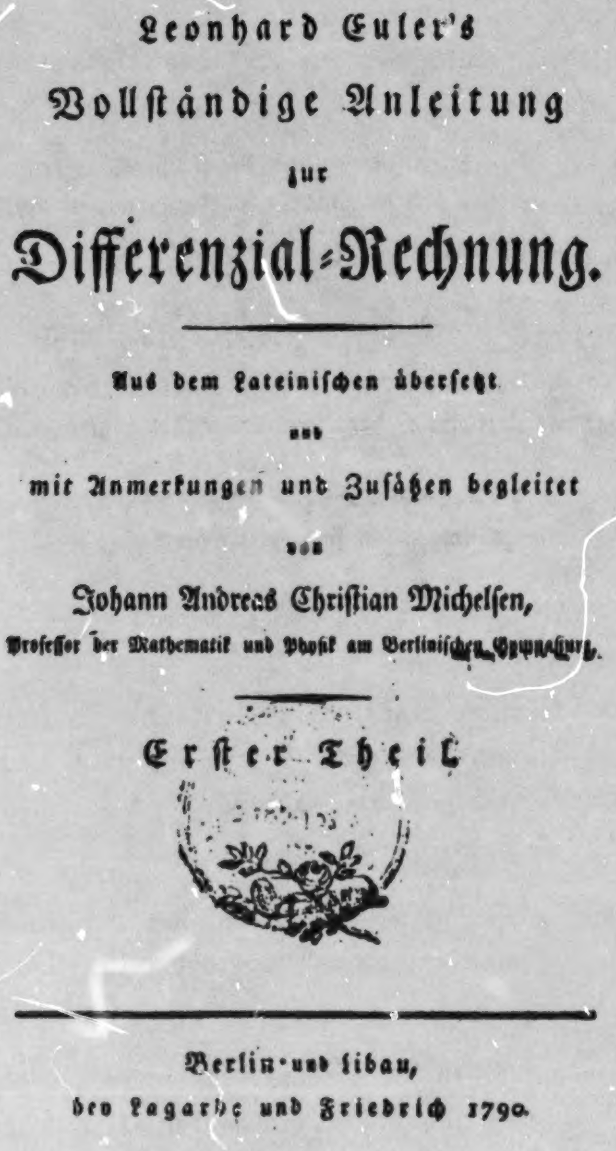
Indeß folgt hieraus nicht, daß die imaginären Größen größer ſeyn als das Unendliche, denn aus der vorhin angeführten Reihe

$$\text{ꝛc.} + \sqrt{-3} + \sqrt{-2} + \sqrt{-1} + 0 + \sqrt{1} + \sqrt{2} + \sqrt{3} + \text{ꝛc.}$$

würde eben ſo gut folgen, daß ſie kleiner wären.

From the first sequence one would include that the imaginary numbers are greater than infinity, and from the second that they are smaller.

In modern terms, we would say that Wallis' argument is invalid, because it assumes implicitly that the sequence $n \longrightarrow \frac{1}{n}$ is continuous, which it clearly is not, as the graph in answer to Qu. 5 shows. There is a discontinuity at 0. The function is decreasing for all n, except at n = 0.



Leonhard Euler's

Vollſtändige Anleitung

zur

Differenzial-Rechnung.

Aus dem Lateiniſchen überſetzt

und

mit Anmerkungen und Zuſätzen begleitet

von

Johann Andreas Chriſtian Michelſen,

Profeſſor der Mathematik und Phyſik am Berliniſchen Gymnaſium.

Erſter Theil.

Berlin und Libau,
bey Lagarde und Friedrich 1790.

Front page of Euler's *Differenzial Rechnung.*

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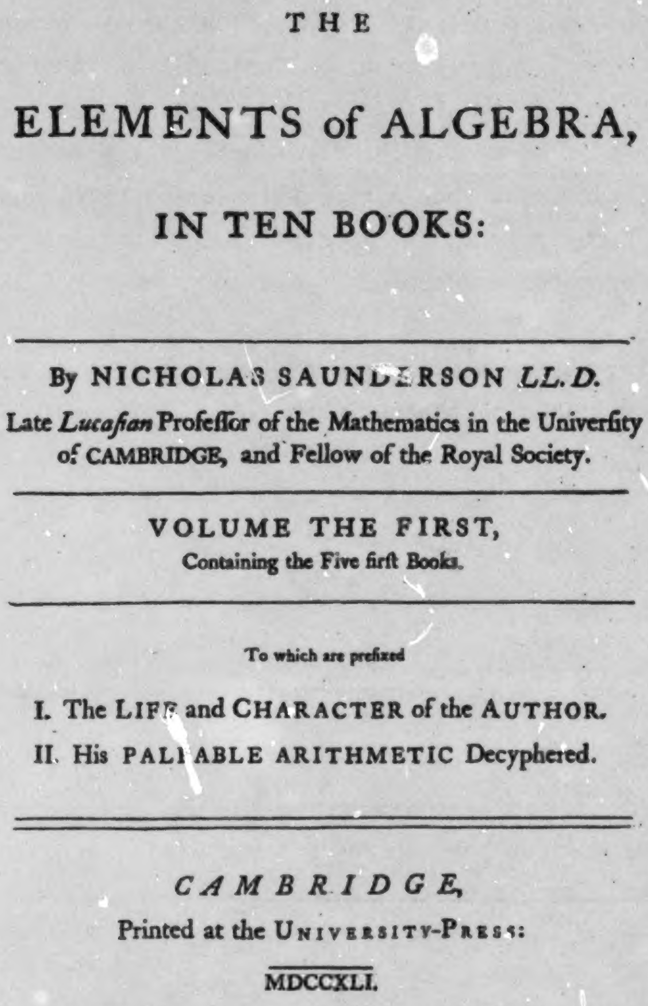
The Negative numbers

Worksheet: Nicholas Saunderson



THE

ELEMENTS of ALGEBRA,

IN TEN BOOKS:

By NICHOLAS SAUNDERSON *LL.D.*
Late *Lucaſian* Profeſſor of the Mathematics in the Univerſity
of CAMBRIDGE, and Fellow of the Royal Society.

VOLUME THE FIRST,
Containing the Five firſt Books.

To which are prefixed

I. The LIFE and CHARACTER of the AUTHOR.
II. His PALPABLE ARITHMETIC Decyphered.

CAMBRIDGE,
Printed at the UNIVERSITY-PRESS:
MDCCXLI.

The English mathematician, Nicholas Saunderson, was born in 1682 and died in 1739. At the age of one year he contracted smallpox and became blind. This tragedy did not, however, prevent him from studying, teaching and living a creative life. He learnt Greek, Latin and French, and "heard" the works of Euclid, Archimedes and Diophantus in the original, learning some parts by heart.

His father taught him arithmetic and he was very skilled in solving problems and executing long calculations, using his prodigious memory and his creation "palpable arithmetic", a nail board for doing arithmetic and creating geometric shapes with silk threads - the forerunner of the modern geoboard. Saunderson made a reputation as an excellent teacher at Cambridge, where he occupied the Lucasian chair, some years previously occupied by Newton.*

The following is an extract from his *The Elements of Algebra,* which was published posthumously in 1741.

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* Further details on the man and his work can be found in the introduction to his book:
Saunderson, N., *The Elements of Algebra,* Vol. 1, Cambridge 1741.

Of the multiplication of algebraic quantities.

And firſt, how to find the ſign of the product in multiplication, from thoſe of the multiplicator and multiplicand given.

5. Before we can proceed to the multiplication of algebraic quantities, we are to take notice, that if the ſigns of the multiplicator and multiplicand be both alike, that is, both affirmative, or both negative, the product will be affirmative, otherwiſe it will be negative: thus + 4 multiplied into + 3, or — 4 into — 3 produces in either caſe + 12: but — 4 multiplied into + 3, or + 4 into — 3 produces in either caſe — 12.

If the reader expects a demonſtration of this rule, he muſt firſt be advertiſed of two things: *firſt*, that numbers are ſaid to be in arithmetical progreſſion, when they increaſe or decreaſe with equal differences, as 0, 2, 4, 6; or 6, 4, 2, 0; alſo as 3, 0, — 3; 4, 0, — 4; 12, 0, — 12; or — 12, 0, + 12: whence it follows, that three terms are the feweſt that can form an arithmetical progreſſion; and that of theſe, if the two firſt terms be known, the third will eaſily be had: thus if the two firſt terms be 4 and 2, the next will be 0; if the two firſt be 12 and 0, the next will be — 12; if the two firſt be — 12 and 0, the next will be + 12, &c.

2dly, If a ſet of numbers in arithmetical progreſſion, as 3, 2 and 1, be ſucceſſively multiplied into one common multiplicator, as 4, or if a ſingle number, as 4, be ſucceſſively multiplied into a ſet of numbers in arithmetical progreſſion, as 3, 2 and 1, the products 12, 8 and 4, in either caſe, will be in arithmetical progreſſion.

This being allowed, (which is in a manner ſelf-evident,) the rule to be demonſtrated reſolves itſelf into four caſes:

1ſt, That + 4 multiplied into + 3 produces + 12.
2dly, That — 4 multiplied into + 3 produces — 12.
3dly, That + 4 multiplied into — 3 produces — 12.
And *laſtly*, that — 4 multiplied into — 3 produces + 12. Theſe caſes are generally expreſſed in ſhort thus: firſt + into + gives +; ſecondly — into + gives —; thirdly + into — gives —; fourthly — into — gives +.

Caſe 1ſt. That + 4 multiplied into + 3 produces + 12, is ſelf-evident, and needs no demonſtration; or if it wanted one, it might receive it from the firſt paragraph of the 3d article; for to multiply + 4 by + 3 is the ſame thing as to add 4 + 4 + 4 into one ſum; but 4 + 4 + 4 added into one ſum give + 12, therefore + 4 multiplied into + 3, gives + 12.

Caſe 2d. And from the ſecond paragraph of the 3d art. it might in like manner be demonſtrated, that — 4 multiplied into + 3 produces — 12: but I ſhall here demonſtrate it another way, thus: multiply the terms of this arithmetical progreſſion 4, 0, — 4, into + 3, and the products will be in arithmetical progreſſion, as above; but the two firſt products are 12 and 0; therefore the third will be — 12; therefore — 4 multiplied into + 3, produces — 12.

Caſe 3d. To prove that + 4 multiplied into — 3 produces — 12; multiply + 4 into + 3, 0, and — 3 ſucceſſively, and the products will be in arithmetical progreſſion; but the two firſt are 12 and 0, therefore the third will be — 12; therefore + 4 multiplied into — 3 produces — 12.

Caſe 4th. Laſtly, to demonſtrate, that — 4 multiplied into — 3 produces + 12, multiply — 4 into 3, 0, and — 3 ſucceſſively, and the products will be in arithmetical progreſſion; but the two firſt products are — 12 and 0, by the ſecond caſe; therefore the third product will be + 12; therefore — 4 multiplied into — 3 produces + 12.

Caſ. 2d. $\begin{array}{rrr} +4, & 0, & -4 \\ +3, & +3, & +3 \\ \hline +12, & 0, & -12. \end{array}$ *Caſ. 3d,* $\begin{array}{rrr} +4, & +4, & +4 \\ +3, & 0, & -3 \\ \hline +12, & 0, & -12. \end{array}$

Caſ. 4th, $\begin{array}{rrr} -4, & -4, & -4 \\ +3, & 0, & -3 \\ \hline -12, & 0, & +12. \end{array}$

Theſe 4 caſes may be alſo more briefly demonſtrated thus: + 4 multiplied into + 3 produces + 12; therefore — 4 into + 3, or + 4 into — 3 ought to produce ſomething contrary to + 12, that is, — 12: but if — 4 multiplied into + 3 produces — 12, then — 4 multiplied into — 3 ought to produce ſomething contrary to — 12, that is, + 12; ſo that this laſt caſe, ſo very formidable to young beginners, appears at laſt to amount to no more than a common principle in Grammar, to wit, that two negatives make an affirmative; which is undoubtedly true in Grammar, though perhaps it may not always be obſerved in languages.

Questions

1. Before giving a demonstration of the various cases of the multiplication rule, Saunderson notes two things. What is the difference between them?

2. Reread the extract and note down all the definitions, assumptions and all the conclusions.

3. What is the difference between the first demonstration of the four cases and the second?

4. In what way are Saunderson's demonstrations different from and in what way similar to, those given in the school text you use?
   (In the Hebrew version, this question is specifically directed at the teaching texts used in the Rehovot programme.)

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The Negative numbers

Answer sheet: Nicholas Saunderson

1. *Firstly,* Saunderson *defines* an arithmetic progression, and gives examples.

   Secondly, he makes a basic *assumption* (axiom), on which he bases his proofs of the various cases. The axiom is that if we multiply an arithmetic progression by a number, or a number by an arithmetic progression, the result is another arithmetic progression.
   In the case where the terms of the A.P. are positive and the multiplier is also positive, we do not really need the assumption, and Saunderson brings an example. But his assumption extends to the case where there are also negative terms.

   Mathematically one may object that his assumption is "as big as" that which he is trying to prove, but *didactically* there is something to be said for it - see answer to question 4.

2.

Definitions	Assumptions	Conclusions
Numbers are said to be in arithmetic progression, when they increase or decrease with equal differences	*If a set of numbers in arithmetical progression be successively multiplied into one common multiplicator, or if a single number be successively multiplied into a set of numbers in arithmetical progression, the products, in either case, will be in arithmetical progression.**	$+3\cdot+4 = +12$ $+3\cdot-4 = -12$ $-3\cdot+4 = -12$ $-3\cdot-4 = +12$
	If we replace one term of a product by its negative (*contrary*), the product also becomes the negative (*contrary*) of the original.	As above

* For the one case of "positive x positive" this is not an assumption, but can be "proved" from earlier work with products of positive numbers. But for all the other cases it is used as an assumption on which to base the "demonstration".

3. As Saunderson himself remarks, the second demonstration is given in brief. We assume that his objective was didactical: to reinforce the result obtained in the first demonstration by looking at it from another point of view.
In the first demonstration, he very carefully states his assumption, and makes explicit its use in each case.
In the second, the assumption (as given in the answer to Qu. 2) is given implicitly as part of the demonstration.

To this he adds the suggestive remark from grammar. (See also answer to Qu. 4.)

4. Clearly the answer depends on the text in use. In the Rehovot programme there are three parallel texts designed to be used with different ability levels. For the top stream the approach is "mathematical" in that the basic assumption is that the basic laws, such as distributively of multiplication over addition, carry over from the set of positive numbers to the set of all numbers. The properties dealt with in the extract are then proved*. There is a qualitative mathematical difference between this approach and that of Saunderson, in that we assume a general mathematical property to hold, rather than a very particular one.

   For the two other streams the approach has some affinity with the first demonstration by Saunderson, in that descending arithmetic sequences of positive numbers are multiplied by a positive number. The sequence is then extended to negative numbers and the students are asked to complete the sequence of products intuitively, and so "discover" the properties of multiplication. By suitable choices of sequences, all the results can be gradually obtained.

The difference between this approach and that of Saunderson is that in the Rehovot texts, the approach is entirely intuitive, whereas Saunderson "demonstrates".

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*In the Worksheet George Peacock (qu. 4) we bring this approach as it appears in SMSG.

Saunderson's second demonstration has no parallel in the Rehovot texts. Didactically it may be regarded as adding a little intuition to the approach to be found in some older texts, in which the bare rules are stated, as if divinely ordained, and then exercised.

NICHOLAS SAUNDERSON LLD
Lucaſian Profeſſor of Mathematicks in
the University of Cambridge
Died 19. Ap. 1739 Aged 56
I. Vanderbank pinx. 171[illegible] From the Original painted for Martin Folkes Esq. G. Vander Gucht Sculp.

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The Negative Numbers

Worksheet: Leonard Euler

It is maintained that Leonard Euler (1707 - 1783) was the most prolific of all mathematicians. Even total blindness in the last seventeen years of his life did not diminish his creativity. He made contributions to analysis (which was then the relatively new "creation" of Newton and Leibniz), algebra, trigonometry, number theory and many other fields.

Euler was born in Basel, and against the wishes of his father, who was a theologian, he turned to mathematics. At the age of 17 he already received a degree in mathematics, and at 19 published his first paper. Following the brothers Bernoulli (part of a family of Swiss mathematicians), he moved to St. Petersburg, where he married and had 13 children. In 1740 he moved to the academy in Berlin, returning to St. Petersburg in 1766.*

Among his many creations are several "textbooks" including the *Elements of Algebra*. This book was written (or better, dictated) by Euler after he went blind and was published in 1770. It was translated into several languages and continued to be published for some 150 years - we have seen an edition published in German during the first world war, and maybe this wasn't the last.
The following extract is taken from an English (fourth) edition published in 1828.

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*For further details on the man and his contribution to mathematics see Boyer, C.B., *A History of Mathematics*. J. Wiley & Sons, 1969.

31. Hitherto we have considered only positive numbers; and there can be no doubt, but that the products which we have seen arise are positive also: viz. $+a$ by $+b$ must necessarily give $+ab$. But we must separately examine what the multiplication of $+a$ by $-b$, and of $-a$ by $-b$, will produce.

32. Let us begin by multiplying $-a$ by 3 or $+3$. Now, since $-a$ may be considered as a debt, it is evident that if we take that debt three times, it must thus become three times greater, and consequently the required product is $-3a$. So if we multiply $-a$ by $+b$, we shall obtain $-ba$, or, which is the same thing, $-ab$. Hence we conclude, that if a positive quantity be multiplied by a negative quantity, the product will be negative; and it may be laid down as a rule, that + by + makes + or *plus*; and that, on the contrary, + by −, or − by +, gives −, or *minus*.

33. It remains to resolve the case in which − is multiplied by −; or, for example, $-a$ by $-b$. It is evident, at first sight, with regard to the letters, that the product will be ab; but it is doubtful whether the sign +, or the sign −, is to be placed before it; all we know is, that it must be one or the other of these signs. Now, I say that it cannot be the sign −: for $-a$ by $+b$ gives $-ab$, and $-a$ by $-b$ cannot produce the same result as $-a$ by $+b$; but must produce a contrary result, that is to say, $+ab$; consequently, we have the following rule: − multiplied by − produces +, that is, the same as + multiplied by + [*].

Questions

1. The multiplication of (real) numbers is divided into four cases

   $+\cdot+$, $+\cdot-$, $-\cdot+$ and $-\cdot-$.

   How does Euler deal with each of these?

2. Find the hidden assumptions in the given extract (there are at least three).

3. The extract is taken from a "textbook". Suppose that you were to use this text in your class, and discuss its strengths and weaknesses.

4. What is the difference between Euler's arguments and Saunderson's ?

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The Negative Numbers

Answer sheet: Leonard Euler

1. i) The case $+\cdot+$ is self evident - "$+a$ by $+b$ must necessarily give $+ab$".

   ii) The two cases $+\cdot-$ and $-\cdot+$ are dealt with together. The explanation uses a model - that of a debt to represent the negative quantity. The model does not distinguish between $+\cdot-$ and $-\cdot+$.

   iii) In the fourth case, $-\cdot-$, "it is evident at first sight" that the product of $-a$ and $-b$ must be of magnitude ab. It remains only to determine whether we should prefix + or -. Euler resolves this by "contradiction": $(-a)\cdot(+b)$ and $(-a)\cdot(-b)$ cannot have the same sign (why Euler does not say), hence the assumption that the latter is equal to $-ab$ leads to a contradiction. Hence $(-a)\cdot(-b) = +ab$.

Thus Euler uses three essentially different types of "argument" to resolve the four cases:

(i) "it is clear that";
(ii) model to suggest the general result;
(iii) mathematical-intuitive argument.

2. Euler's assumptions are all implicit:
   - i) Transfer from the algebraic problem to a model (debt) and back again, as a legitimate process in reaching a mathematical conclusion.
   - ii) The extension of the commutative law, from multiplication of positive numbers to that of negative numbers. Thus Euler exemplifies only "negative times positive".
   - iii) The assumption that the numerical value of $(-a)\cdot(-b)$ must be ab.
   - iv) The assumption that $(-a)\cdot(+b)$ and $(-a)\cdot(-b)$ cannot have the same sign. This assumption itself may have been tacitly based on a more fundamental assumption, i.e. that the *cancellation law* can be extended from the positive to the negative numbers. Thus if we assume that

$$(-a)\cdot(+b) = (-a)\cdot(-b)$$

and that the cancellation law holds, this implies that

$$+b = -b$$

which is false if $b \neq 0$.

3. Various points arise in this extract:
   - i) The section is didactically well organised. There is an introduction, from an example, the general case is "deduced". At the end of each sub-section there is an interim summary, and each new case is introduced.

ii) We have already remarked in the answer to Qu. 2 on some assumptions - the didactical validity of these assumptions is dependent on the mathematical maturity of the students.

Another point arising is the fact that Euler (in his extract) regards the variable a as representing a positive quantity. (Histories of mathematics credit the Dutch mathematician, Johann Hudde (1629-1704) with being the first to use a letter to represent a positive or negative quantity.) There is a "didactical danger" to the present day in this, since students may tend to assume in other contexts that a always represents a positive quantity and that $-a$ is always negative.

iii) Euler writes: "... since -a may be considered as a debt, it is evident that if we take the debt three times, it must become three times greater, and consequently the required product is -3a".

This may well give the impression that -3a is greater than -a, which is not true (if a is positive). The order relation in the negative numbers, e.g. that -3 is bigger than -4, often causes difficulty, and one should avoid possibilities of reinforcing false impressions.

iv) In short tests on negative numbers given to some grade 7 classes we found an interesting (if not altogether serious) mistake. A few students justified

$$(-4)\cdot(+3) = (+4)\cdot(-3)$$

by recourse to the commutative law.

This may well arise from our (and Euler's) imprecise (if suggestive) phrasing in such results as

"... + by - , or - by + , gives - " .

It would be presumptive if we were to leave the impression that we are criticising Euler. In this question, we have used the text as an object on which to having a discussion of some didactical points arising in the introduction of the multiplication of negative numbers. Most treatments have their pros and cons and some compromise is usually necessary. It is, however, of more than academic interest to be aware of any mistakes we may be inducing by our teaching, in order that we may plan counter action.

As far as Euler is concerned, we should remember that his text was very popular and went through many editions, in English, French, Russian and German. His text belongs to an era of algebra textbooks, as described in the following*:

---

* Boyer C.B., *A History of Mathematics*, 1968, Wiley & sons.

The leading Continental mathematicians of the mid-eighteenth century were primarily analysts, but we have seen that their contributions were not limited to analysis. D'Alembert had given an imperfect proof of the fundamental theorem of algebra, and Clairaut in 1740 had published a textbook, *Élémens d'algèbre*, which was so popular it went through a sixth edition in 1801. Euler not only contributed to the theory of numbers, but also composed a popular algebra textbook that appeared in German and Russian editions at St. Petersburg in 1770–1772, in French (under the auspices of d'Alembert) in 1774, and in numerous other versions, including American editions in English. The exceptionally didactic quality of Euler's *Algebra* is attributed to the fact that it was dictated by the blind author through a relatively untutored domestic. The textbooks of Clairaut and Euler were not widely used in England, in part because of British mathematical isolationism during the later eighteenth century and in part because Maclaurin and others had composed good textbooks on an elementary level. Maclaurin's *Treatise of Algebra* went through half a dozen editions from 1748 to 1796. A rival *Treatise of Algebra* by Thomas Simpson (1710–1761) boasted at least eight editions at London from 1745 to 1809; another, *Elements of Algebra*, by Nicholas Saunderson (1682–1739), enjoyed five editions between 1740 and 1792. Simpson was a self-taught genius who won election to the Royal Society in 1745 but whose turbulent life ended in failure half a dozen years later. His name nevertheless is preserved in the so-called Simpson's rule, published in his *Mathematical Dissertations on Physical and Analytical Subjects* (1743), for approximate quadratures using parabolic arcs; but this result had appeared in somewhat different form in 1668 in the *Exercitationes geometricae* of James Gregory. Saunderson's life, by contrast, was an example of personal triumph over an enormous handicap—total blindness from the age of one, resulting from an attack of smallpox.

Algebra textbooks of the eighteenth century illustrate a tendency toward increasingly algorithmic emphasis, while at the same time there remained considerable uncertainty about the logical bases for the subject. Most authors felt it necessary to dwell at length on the rules governing multiplications of negative numbers, and some rejected categorically the possibility of multiplication of two negative numbers. The century was, par excellence, a textbook age in mathematics, and never before had so many books appeared in so many editions.

4. In his first approach, Saunderson deals with each of the four cases separately, deducing each one (except, possibly, the first) from his basic assumption that a number multiplied by an arithmetic progression, or an arithmetic progression by a number, gives an arithmetic progression. Euler, on the other hand, makes a number of assumptions (as detailed in the answer to Qu. 2), using the "most convenient" in each case.

   In his second approach, Saunderson again uses a single argument, which is similar to that of Euler in the case $-\cdot-$. In this connection, it is interesting to quote Smith*, who remarks that in the "old textbooks" there is a great effort to "prove" the "law of signs", and then continues:

> ...The favorite one of these "proofs" was this: Since multiplying $-b$ by a gives $-ab$, therefore if the sign of the multiplier is changed, of course the sign of the product must also be changed. As a proof, it is like saying that if A, a white man, wears black shoes, therefore it follows that B, a black man, must wear shoes of an opposite color.

---

* Smith, D.E. *The Teaching of Elementary Mathematics*, Macmillan Co., 1900, p.188.

Smith also wrote a popular two volume history of mathematics.

ELEMENTS OF ALGEBRA,

BY

LEONARD EULER,

TRANSLATED FROM THE FRENCH;

WITH THE

NOTES OF M. BERNOULLI, &c.

AND THE

ADDITIONS OF M. DE LA GRANGE.

FOURTH EDITION,
CAREFULLY REVISED AND CORRECTED.

BY THE REV. JOHN HEWLETT, B.D. F.A.S. &c.

TO WHICH IS PREFIXED,

A Memoir of the Life and Character of Euler,

BY THE LATE

FRANCIS HORNER, ESQ., M. P

LONDON:
PRINTED FOR LONGMAN, REES, ORME, AND CO.
PATERNOSTER-ROW.
1828.

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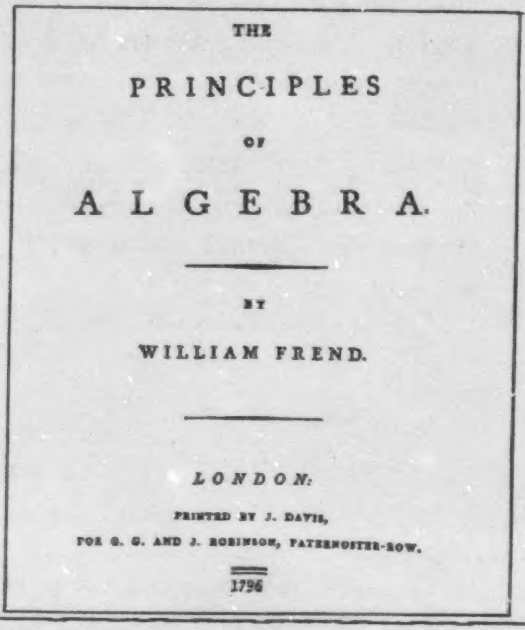
The Negative numbers

Worksheet: The last of the opposition - W. Frend

The following extracts are taken from *The Principles of Algebra* by William Frend (1757-1841), published in 1796.

Frend, an Englishman, was a rebel in more senses than one, and was banished from Cambridge where he had been a tutor*. He was also the father-in-law of a much more famous mathematician, Augustus De Morgan, whom we shall quote in the answer sheet.



THE
PRINCIPLES
OF
ALGEBRA.

BY
WILLIAM FREND.

LONDON:
PRINTED BY J. DAVIS,
FOR G. G. AND J. ROBINSON, PATERNOSTER-ROW.
1796

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* For further details on Frend see, for example De Morgan A., *A Budget of Paradoxes, Vol. 1*, Open Court, 1915.

...The firſt error in teaching the principles of algebra is obvious on peruſing a few pages only in the firſt part of Maclaurin's Algebra. [*]Numbers are there divided into two ſorts, poſitive and negative; and an attempt is made to explain the nature of negative numbers, by alluſions to book-debts and other arts. Now, when a perſon cannot explain the principles of a ſcience without reference to metaphor, the probability is, that he has never thought accurately upon the ſubject. A number may be greater or leſs than another number; it may be added to, taken from, multiplied into, and divided by another number; but in other reſpects it is very untractable: though the whole world ſhould be deſtroyed, one will be one, and three will be three; and no art whatever can change their nature. You may put a mark before one, which it will obey: it ſubmits to be taken away from another number greater than itſelf, but to attempt to take it away from a number leſs than itſelf is ridiculous. Yet this is attempted by algebraiſts, who talk of a number leſs than nothing, of multiplying a negative number into a negative number and thus producing a poſitive number, of a number being imaginary.

...This is all jargon, at which common ſenſe recoils; but, from its having been once adopted, like many other figments, it finds the moſt ſtrenuous ſupporters among thoſe who love to take things upon truſt, and hate the labour of a ſerious thought.

---

[*] Colin Maclaurin (1698-1746), a student of Newton, made contributionsto Analysis. About Maclaurin's Algebra, among other "textbooks" published in the 18th century, see the answer sheet: Euler.

Questions

1. Summarise Frend's arguments against the use of negative numbers as expressed in the two extracts.

2. How ~~should~~ would you answer him?

Having seen Frend's general attitude to the subject of negative numbers, we now look at his treatment of certain algebraic topics.

ON EQUATIONS OF THE SECOND ORDER.

AN equation having, in any term, an unknown quantity of the ſecond power, and no higher power in any other term, is called an equation of the ſecond order, and all equations of this order are reduceable by the preceding rules to the following forms.

1. $x^2 = b$
2. $x^2 + ax = b$
3. $x^2 - ax = b$
4. $ax - x^2 = b$.

Questions

3. a) Does Frend, in fact, include all possible equations of second order in his representation?
   (Remember that *a* and *b* represent positive numbers only.)

   b) In which of the four cases may Frend run into a negative root?

   c) What do you consider to be the disadvantages of this approach?

The following extract is an example of how Frend solves a second order equation.

(111)

Let $x + \sqrt{5x + 10} = 8$.

$$\therefore \sqrt{5x + 10} = 8 - x$$

$$\therefore 5x + 10 = 64 - 16x + x^2$$

$$\therefore 5x + 16x - x^2 = 64 - 10$$

$$\therefore 21x - x^2 = 54 \qquad \text{(a)}$$

$$\therefore \frac{441}{4} - 21x + x^2 = \frac{441}{4} - 54 = \frac{441 - 216}{4} = \frac{225}{4} \qquad \text{(b)}$$

$$\therefore \sqrt{\frac{441}{4} - 21x + x^2} = \sqrt{\frac{225}{4}}$$

$$\frac{21}{2} - x = \frac{15}{2}$$

$$\therefore x = \frac{21}{2} - \frac{15}{2} = \frac{6}{2} = 3.$$

In this caſe the root $x - \frac{21}{2}$ cannot be uſed; for, by the conditions of the queſtion, x is leſs than eight.

Questions

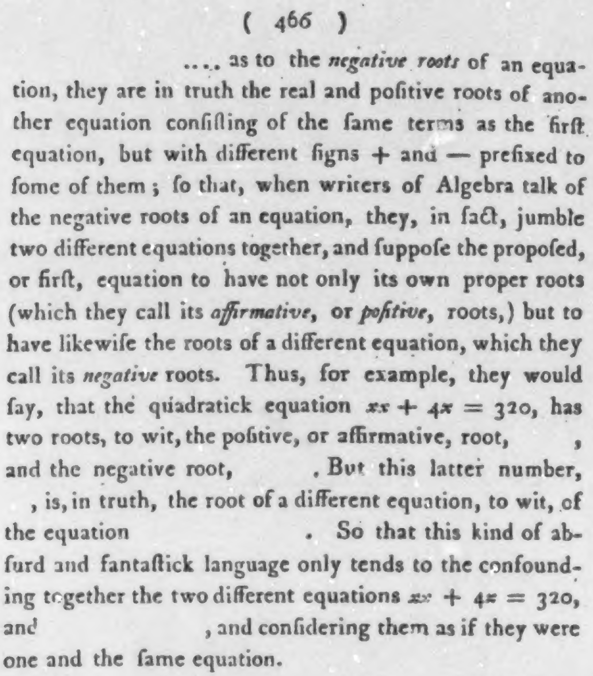
4. a) Explain the step from (a) to (b).

   b) Why does Frend maintain that "x is less than eight"?

   c) Would you also reject the second root?

   d) If Frend had begun the question at the point (a), would his treatment have been the same?

In the previuos sections we have seen how Frend deals with equations of second order. In the following extract (taken from an appendix to Frend's book, written by Francis Maseres (1731-1824), an Englishman and member of the Royal Society) we see how the "negative roots" obtained by other mathematicians are explained. Note that we have deliberately ommited some bits of the text - see Qu. 5.



(466)

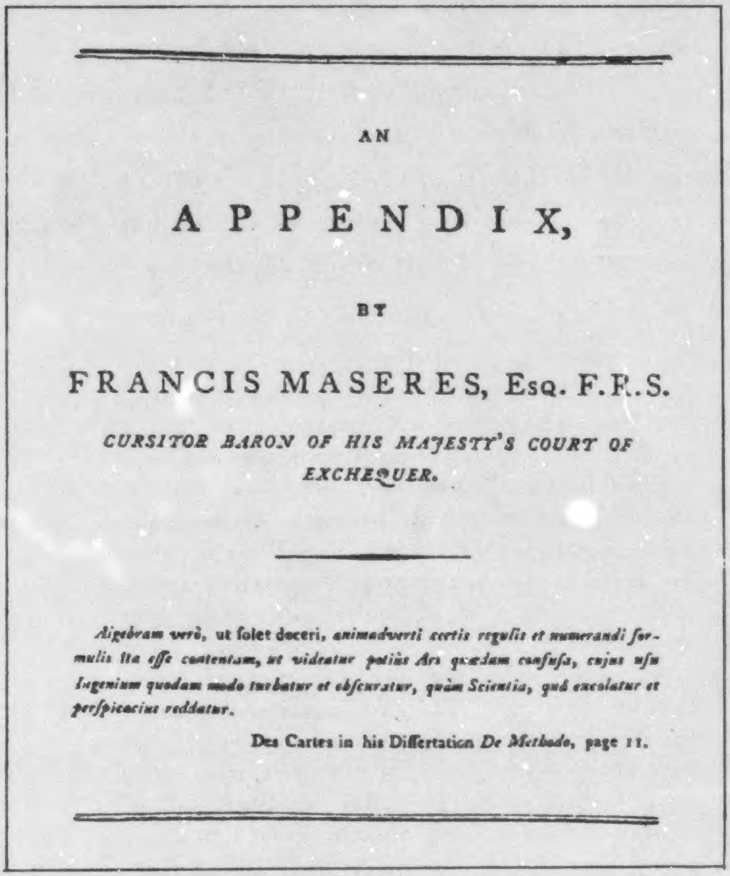
.... as to the *negative roots* of an equation, they are in truth the real and poſitive roots of another equation conſiſting of the ſame terms as the firſt equation, but with different ſigns + and — prefixed to ſome of them; ſo that, when writers of Algebra talk of the negative roots of an equation, they, in fact, jumble two different equations together, and ſuppoſe the propoſed, or firſt, equation to have not only its own proper roots (which they call its *affirmative*, or *poſitive*, roots,) but to have likewiſe the roots of a different equation, which they call its *negative* roots. Thus, for example, they would ſay, that the quadratick equation $xx + 4x = 320$, has two roots, to wit, the poſitive, or affirmative, root, , and the negative root, . But this latter number, , is, in truth, the root of a different equation, to wit, of the equation . So that this kind of abſurd and fantaſtick language only tends to the confounding together the two different equations $xx + 4x = 320$, and , and conſidering them as if they were one and the ſame equation.

Questions

5. a) Fill in the bits we have omitted in the text.

   b) Apply the above remarks to $x^2 - 5x = 50$ and $4x - x^2 = 3$.

   c) Does this extract remind you of previous work in this series of worksheets?



AN

APPENDIX,

BY

FRANCIS MASERES, Esq. F.R.S.

CURSITOR BARON OF HIS MAJESTY'S COURT OF EXCHEQUER.

> *Algebram verò,* ut ſolet doceri, *animadverti certis regulis et numerandi formulis ita eſſe contentam, ut videatur potiùs Ars quædam confuſa, cujus uſu Ingenium quodam modo turbatur et obſcuratur, quàm Scientia, quâ excolatur et perſpicacius reddatur.*
>
> Des Cartes in his Diſſertation *De Methodo*, page 11.

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The Negative numbers

Answer sheet: The last of the opposition - W. Frend

1. Frend's opposition to the use of negative numbers is based on a number of points. In the first extract we have the following:

    i) The explanations of negative numbers uses examples from outside mathematics -

    > ...an attempt is made to explain the nature of negative numbers, by alluſions to book-debts and other arts. Now, when a perſon cannot explain the principles of a ſcience without reference to metaphor, the probability is, that he has never thought accurately upon the ſubject.

    ii) A number is a number - its essential nature is not changed by having "a mark" put before it.

    iii) A number "will obey" a mark put before it - as long as it makes sense to do so. Thus

    > ...it ſubmits to be taken away from another number greater than itſelf, but to attempt to take it away from a number leſs than itſelf is ridiculous.
    >
    > ...This is all jargon, at which common ſenſe recoils

In the second extract we have the following points:

iv) The expression $-f$ makes no sense, because the mark — before *f* denotes that it is to be taken away from ſome number which is not written down,

v) The marks (symbols) without numbers make no sense. Thus - x - = + is nonsense both in algebra and in everyday language.

2. The following two extracts, written by two famous English mathematicians, who both were also skilled writers on mathematics, can be used as the basis for answering Frend.

The first extract is taken from Whitehead, A.N., *An Introduction to Mathematics,* first published in the Everyman series of texts in the early part of the century. Whitehead (1861-1947), with Bertrand Russell (1872-1970) made a considerable contribution to the foundations and philosophy of mathematics. The work in question is an extremely readable popularisation of elementary mathematics.

Consider the statement $2+3=5$. We add 3 to 2 and obtain 5. Think of the operation of adding 3: let this be denoted by $+3$. Again $4-3=1$. Think of the operation of subtracting 3: let this be denoted by -3. Thus instead of considering the real numbers in themselves, we consider the *operations* of adding or subtracting them: instead of $\sqrt{2}$, we consider $+\sqrt{2}$ and $-\sqrt{2}$, namely the operations of adding $\sqrt{2}$ and of subtracting $\sqrt{2}$. Then we can add these operations, of course in a different sense of addition to that in which we add numbers. The sum of two operations is the single operation which has the same effect as the two operations applied successively. In what order are the two operations to be applied? The answer is that it is indifferent, since for example

$$2+3+1=2+1+3;$$

so that the addition of the steps $+3$ and $+1$ is commutative.

Mathematicians have a habit, which is puzzling to those engaged in tracing out meanings, but is very convenient in practice, of using the same symbol in different though allied senses. The one essential requisite for a symbol in their eyes is that, whatever its possible varieties of meaning, the formal laws for its use shall always be the same. In accordance with this habit the addition of operations is denoted by + as well as the addition of numbers. Accordingly we can write

$$(+3)+(+1)=+4;$$

where the middle + on the left-hand side denotes the addition of the operations $+3$ and $+1$. But, furthermore, we need not be so very pedantic in our symbolism, except in the rare instances when we are directly tracing meanings; thus we always drop the first + of a line and the brackets, and never write two + signs running. So the above equation becomes

$$3+1=4 \quad \ldots$$

What is the use of all this elaboration? ...

The answer is that what the mathematician is seeking is Generality. ...

Any limitation whatsoever upon the generality of theorems, or of proofs, or of interpretation is abhorrent to the mathematical instinct. ...

Let us see how generality is gained by the introduction of this idea of operations. Take the equation $x+1=3$; the solution is $x=2$. Here we can interpret our symbols as mere numbers, and the recourse to "operations" is entirely unnecessary. But, if x is a mere number, the equation $x+3=1$ is nonsense. For x should be the number of things which remain when you have taken 3 things away from 1 thing; and no such procedure is possible. At this point our idea of algebraic form steps in, itself only generalization under another aspect. We consider, therefore, the general equation of the same form as $x+1=3$. This equation is $x+a=b$, and its solution is $x=b-a$. Here our difficulties become acute; for this form can only be used for the numerical interpretation so long as b is greater than a, and we cannot say without qualification that a and b may be any constants. In other words we have introduced a limitation on the variability of the "constants" a and b, which we must drag like a chain throughout all our reasoning. Really prolonged mathematical investigations would be impossible under such conditions. Every equation would at last be buried under a pile of limitations. But if we now interpret our symbols as "operations," all limitation vanishes like magic. The equation $x+1=3$ gives $x=+2$, the equation $x+3=1$ gives $x=-2$, the equation $x+a=b$ gives $x=b-a$ which is an operation of addition or subtraction as the case may be.

The second quotation is taken from De Morgan, A., *A Budget of Paradoxes,* Open Court, 1915. Augustus De Morgan died in 1871. The year of his birth can be found from the puzzle he himself proposed - "I was x years old in the year x^2". He studied at Cambridge, where he was tutored by George Peacock - who will be the subject of another worksheet in this series. Later, he became professor of mathematics at University College, London. De Morgan married Frend's daughter, which did not prevent him disagreeing with Frend's views on mathematics.

De Morgan was a prolific author, and his works are always interesting.

The *Budget of Paradoxes* is, as its name suggests, a collection of paradoxes, but the word does not have its usual meaning - De Morgan writes - *"I use the word in the old sense: a paradox is something which is apart from general opinion, either in subject matter, method, or conclusion."*

Frend is clearly a paradoxer under this definition - since, in his time, there were few serious opponents of negative numbers left.
The following is the quotation. It includes a passage from Maseres*, who also belonged to the "negative paradoxers", and De Morgan's response.

> Maseres speaks as follows: "A single quantity can never be marked with either of those signs, or considered as either affirmative or negative; for if any single quantity, as b, is marked either with the sign + or with the sign — without assigning some other quantity, as a, to which it is to be added, or from which it is to be subtracted, the mark will have no meaning or signification: thus if it be said that the square of —5, or the product of —5 into —5, is equal to +25, such an assertion must either signify no more than that 5 times 5 is equal to 25 without any regard to the signs, or it must be mere nonsense and unintelligible jargon.

---

* Maseres, F., *A Dissertation on the use of the Negative Sign in Algebra ...*, London, 1758.

To which De Morgan replies:

> ... The great difficulty of the opponents of algebra lay in want of power or will to see extension of terms. ... One of my paradoxers was present at a meeting of the Royal Society (in 1864, I think) and asked permission to make some remarks upon a paper. He rambled into other things, and, naming me, said that I had written a book in which two sides of a triangle are pronounced *equal* to the third.[5] So they are, in the sense in which the word is used in complete algebra; in which A+B=C makes A, B, C, three sides of a triangle, and declares that going over A and B, one after the other, is equivalent, in change of place, to going over C at once. My critic, who might, if he pleased, have objected to extension, insisted upon reading me in unextended meaning.

---

[5] The paper was probably one on complex numbers, or possibly one on quaternions, in which direction as well as absolute value is involved.

The extracts emphasize the concept of extension and the search for generality in mathematics. They are fundamental in understanding mathematical activity in general, and in answering Frend in particular.

Specific replies to the points made by Frend may include the following:

i) The use of an example from outside mathematics, he describes as "The first error in teaching the principles of algebra". But it is just in the teaching of mathematics, that a physical or other example can be useful, either to motivate or to reinforce (cf. Saunderson's remark about the double negative in grammar). Clearly, mathematics demands something more rigorous in the end, but examples and models are useful didactical strategies.

Whitehead suggests a mathematical interpretation for -5 as the *operation* of subtracting 5, and then defines what he means by adding and subtracting these operations themselves, thus indulging in a number of "extensions of terms".
It is worth noting that it is not that easy to extend Whitehead's operations to multiplication of negative numbers.

ii) Frend's conception of number (as for many of his contemporaries) is that of a physical magnitude, and he is unwilling to extend. The search for extension and generality, as already remarked, are the essence of mathematical activity. Thus Frend would oppose much progress in mathematics.
The problem with answering Frend is that we are talking about undefined entities, and as long as we do not have an explicit definition of the negative numbers, we cannot argue with complete objectivity.

The mathematician, C.L. Dodgson (1832-1898), better known as Lewis Carroll, made Humpty Dumpty say:

> *When I use a word it means just what I choose it to mean - neither more nor less.**

Here, as elsewhere, Carroll's mathematical background is not far under the surface. A mathematician may define a term to mean what he wishes as long as it is consistent with previous definitions and results.

---

* Carroll L., *The Annotated Alice, with Introduction and notes by Martin Gardner.* Meridian, 1960, p. 269.

The apparent inconsistencies of Wallis, etc. had to be dealt with, but Frend's objection is more a matter of prejudice than logic.

iii) If we extend the concept of number to include negative numbers, then we can also subtract a greater from a smaller. Frend's difficulty is caused, as in ii), by his refusal to extend the number concept.

iv) In the second extract, Frend's problems are compounded by his further unwillingness to allow the minus sign to have any other meaning than subtraction, and that only as in iii). In writing -f, we are giving the sign -, the meaning of "negative", rather than subtraction; or, as Whitehead interprets it, as an operation. Again, we need the extension principle, of attaching new or extended meaning to familiar symbols.
De Morgan's + example, is another illustration of this.

In this connection it is interesting to note that some authors and texts distinguish between the two roles of the minus and other signs. For example, instead of

$$(-4) - (-3)$$

they write $({}^{-}4) - ({}^{-}3)$.

v) An abbreviation is admissible if it does not lead to misunderstanding. The use of " - x - = + " is merely as an abbreviation for the concept "a negative number times a negative number gives a positive number". We have already remarked on a possible didactical difficulty in the Euler answer sheet, but, in general, it is certainly acceptable as a convenient shorthand. Frend's objection was probably fired by over enthusiasm for his cause.

Although, the "true mathematical" explanation of negative numbers troubled many at the time of Frend, few serious opponents of negative numbers were left. The following quotation* illustrates this:

> *Although a study of British algebra textbooks and articles published in the leading British scientific journals of the late 18th and early 19th centuries demonstrates that the problem of negative numbers was a major concern of British thinkers of the period, it also shows that most British mathematicians were reluctant to renounce the riches of algebra because of foundational problems. Although admitting the absence of a satisfactory definition of the negatives, most argued in favor of their retention, if only because of practical considerations. Thinkers who concurred with Maseres and Frend in recognition of the problem but not in their solution included William Greenfield, an amateur mathematician and a professor of rhetoric at the University of Edinburgh,...*

---

* Pycior H., George Peacock and the British Origins of Symbolical Algebra, *Historia Mathematica*, 1981, Vol. 8, p. 23-45.

Greenfield said*:

> ...The truth is, that the whole buſineſs of algebra might be carried on without the conſideration of the negative roots. The difference between ſuch a ſyſtem and the preſent, is preciſely this; that wherever a problem required us to conſider any of the quantities, as exiſting in oppoſite ſituations; wherever, for inſtance, a line or a point was to be conſidered, as ſituated firſt on the right hand, and then on the left; it would be neceſſary, to find and to reſolve a ſeparate equation for each of theſe caſes. Thus, in the analyſis of any particular curve, it would be neceſſary to have a ſeparate equation for each of the four angles of the co-ordinates; except, indeed, the axes were ſo choſen, as to make us certain that there were ſome of theſe angles, in which no part of the curve was to be found. Since, therefore, the uſe of negative quantities frees us from this inconvenience, which, in many caſes, particularly in the analyſis of curves, would be exceedingly perplexing; and ſince it evidently affords ſo great elegance and univerſality to algebraical ſolutions; to find our author gravely declaring that he can ſee no advantage in it, is perfectly aſtoniſhing: As it is to be lamented, that he did not exert his induſtry and ingenuity, rather to confirm than to deſtroy; rather to demonſtrate, how far we might rely on the method of negative quantities, than to overturn at once ſo great a part of the labours of the modern algebraiſts.

Summing up, this period in the history of negative numbers can be considered as an example of one of Wilder's**'laws' governing the evolution of mathematical concepts: *The admissibility and acceptance of a concept will be decided by its degree of fruitfulness. In particular, a concept will not be forever rejected because of its origin or on the grounds of metaphysical criteria such as "unreality".*

---

* In his address to the *Royal Society of Edinburgh (Trans.)*, 1788, Vol. 1, p. 131-145.

** Wilder R.L., *Evolution of Mathematical Concepts*, Transworld Student Library, 1973, p. 197-200.

3. a) The equations given by Frend include those cases only in which at least one root is positive. He does not include the equation

$$x^2 + ax + b = 0 \quad (a, b > 0)$$

because it has two negative (or imaginary) roots.

b) In the first three of Frend's cases one may obtain a negative root.

c) The fact that Frend has to consider four cases, instead of the one

$$ax^2 + bx + c = 0$$

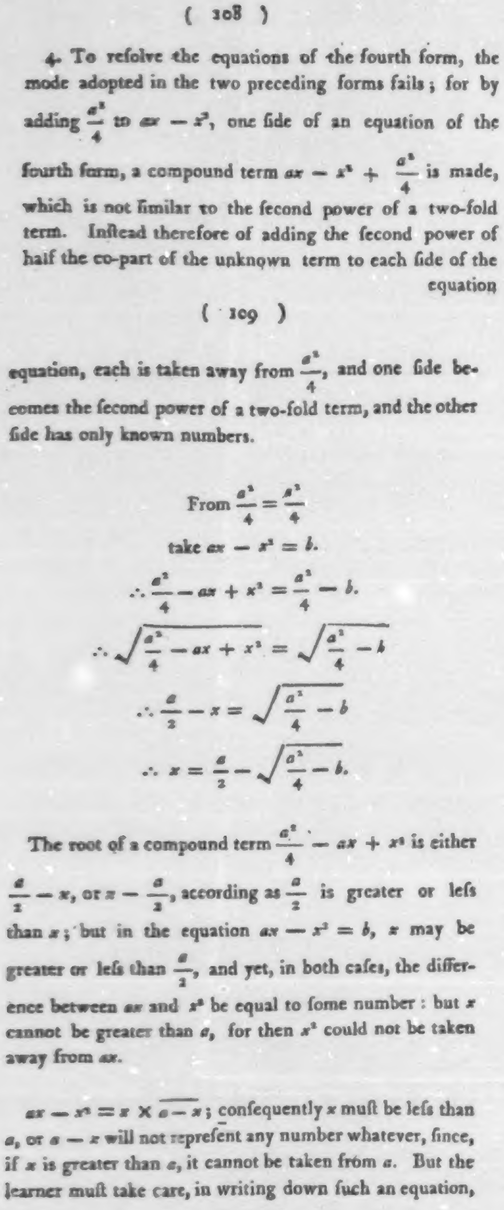
in which a, b and c are unrestricted (real) numbers, is a considerable disadvantage. He discusses the solution of each of the cases separately. The student has to fit any given problem to the appropriate one of the four cases, involving unnecessary technique and memorisation.

4. a) The step from (a) to (b) is the standard method of completing the square. It is worthy of note that Frend "subtracts (a)" from $\frac{441}{4}$, to obtain the complete square. In this step, he uses the principle

$$A - (B - C) = A - B + C .$$

This does not imply any recognition of negative numbers, since Frend is careful to stipulate that $A > B$ and $B > C$. This rule of subtraction was already known even by those who did not deal with negatives at all (see also the Viète work- and answer sheets).

We give below the full text, in which Frend discusses the general case, of which the given problem is an example.



(108)

4. To reſolve the equations of the fourth form, the mode adopted in the two preceding forms fails; for by adding $\frac{a^2}{4}$ to $ax - x^2$, one ſide of an equation of the fourth form, a compound term $ax - x^2 + \frac{a^2}{4}$ is made, which is not ſimilar to the ſecond power of a two-fold term. Inſtead therefore of adding the ſecond power of half the co-part of the unknown term to each ſide of the equation

equation

(109)

equation, each is taken away from $\frac{a^2}{4}$, and one ſide becomes the ſecond power of a two-fold term, and the other ſide has only known numbers.

$$\text{From } \frac{a^2}{4} = \frac{a^2}{4}$$

$$\text{take } ax - x^2 = b.$$

$$\therefore \frac{a^2}{4} - ax + x^2 = \frac{a^2}{4} - b.$$

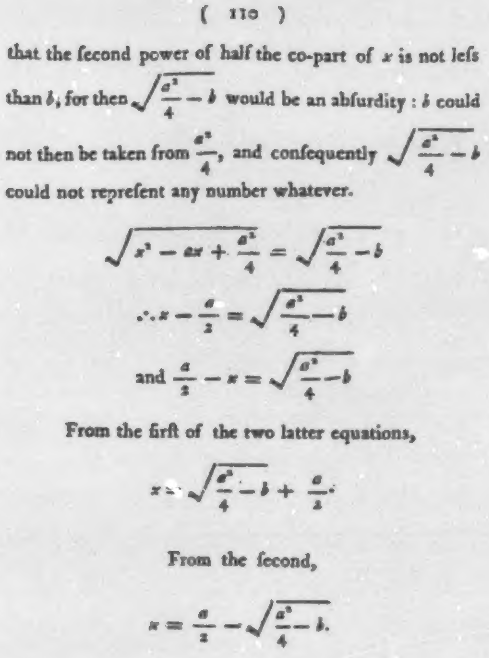
$$\therefore \sqrt{\frac{a^2}{4} - ax + x^2} = \sqrt{\frac{a^2}{4} - b}$$

$$\therefore \frac{a}{2} - x = \sqrt{\frac{a^2}{4} - b}$$

$$\therefore x = \frac{a}{2} - \sqrt{\frac{a^2}{4} - b}.$$

The root of a compound term $\frac{a^2}{4} - ax + x^2$ is either $\frac{a}{2} - x$, or $x - \frac{a}{2}$, according as $\frac{a}{2}$ is greater or leſs than x; but in the equation $ax - x^2 = b$, x may be greater or leſs than $\frac{a}{2}$, and yet, in both caſes, the difference between ax and x^2 be equal to ſome number: but x cannot be greater than a, for then x^2 could not be taken away from ax.

$ax - x^2 = x \times \overline{a - x}$; conſequently x muſt be leſs than a, or $a - x$ will not repreſent any number whatever, ſince, if x is greater than a, it cannot be taken from a. But the learner muſt take care, in writing down ſuch an equation,



(110)

that the ſecond power of half the co-part of x is not leſs than b, for then $\sqrt{\frac{a^2}{4} - b}$ would be an abſurdity : b could not then be taken from $\frac{a^2}{4}$, and conſequently $\sqrt{\frac{a^2}{4} - b}$ could not repreſent any number whatever.

$$\sqrt{x^2 - ax + \frac{a^2}{4}} = \sqrt{\frac{a^2}{4} - b}$$

$$\therefore x - \frac{a}{2} = \sqrt{\frac{a^2}{4} - b}$$

$$\text{and } \frac{a}{2} - x = \sqrt{\frac{a^2}{4} - b}$$

From the firſt of the two latter equations,

$$x = \sqrt{\frac{a^2}{4} - b} + \frac{a}{2}.$$

From the ſecond,

$$x = \frac{a}{2} - \sqrt{\frac{a^2}{4} - b}.$$

b) Given that Frend does not recognize negative quantities, the first equation

$$x + \sqrt{5x + 10} = 8$$

implies that both x and $\sqrt{5x + 10}$ are less than 8.

c) The second root is x = 18, which Frend rejects as unsuitable because of "the conditions of the question". But, quite independently of Frend's particular standpoint, 18 does not satisfy the original equation. It arises in the third line, where both sides of the equation in line two, have been squared. Not only

$$\sqrt{5x + 10} = 8 - x$$

leads to line 3, but also

$$-\sqrt{5x+10} = 8 - x$$

and 18, is the solution of the latter.

In the modern texts the $\sqrt{\ }$ symbol is usually taken to mean the positive root only, but some use it as equivalent to the exponent ½ , i.e. $\sqrt{x} = x^{\frac{1}{2}}$, in in which case, both roots of x are to be understood. In Frend, of course, no such ambiguity can arise. However, he might have made some capital out of the fact that the admission of negative numbers, even implicitly, could lead to ridiculous results such as $18 + 10 = 8$.

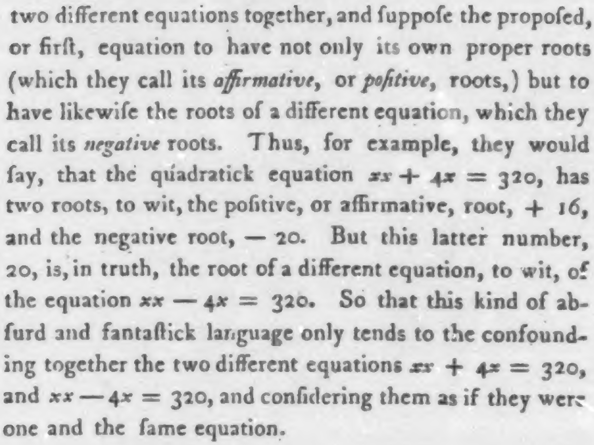
d) Starting from

$$21x - x^2 = 54 ,$$

he would procede as before, exemplifying the fourth of his cases, as brought at the end of the solution of part a). And this time both roots 3 and 18, are admissible, since they are both positive and each step in the argument involves at most subtraction of a smaller quantity from a greater one.

5. a) The "unexpurgated" text reads as follows:

> as to the *negative roots* of an equation, they are in truth the real and poſitive roots of another equation conſiſting of the ſame terms as the firſt equation, but with different ſigns + and — prefixed to ſome of them; ſo that, when writers of Algebra talk of the negative roots of an equation, they, in fact, jumble

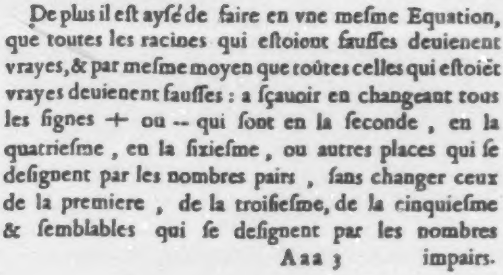


two different equations together, and ſuppoſe the propoſed, or firſt, equation to have not only its own proper roots (which they call its *affirmative*, or *poſitive*, roots,) but to have likewiſe the roots of a different equation, which they call its *negative* roots. Thus, for example, they would ſay, that the quadratick equation $xx + 4x = 320$, has two roots, to wit, the poſitive, or affirmative, root, $+ 16$, and the negative root, $— 20$. But this latter number, 20, is, in truth, the root of a different equation, to wit, of the equation $xx — 4x = 320$. So that this kind of abſurd and fantaſtick language only tends to the confounding together the two different equations $xx + 4x = 320$, and $xx — 4x = 320$, and conſidering them as if they were one and the ſame equation.

b) The equation $x^2 - 5x = 50$ has two roots, 10 and -5. Maseres would claim that the -5 indicates that 5 is the root of another equation $x^2 + 5x = 50$, with the same terms but a change of sign.

There are no problems with $4x - x^2 = 3$, since it has two positive roots, 1 and 3.

c) Maseres is essentially using the result we met in the Descartes worksheet:



De plus il eſt ayſé de faire en vne meſme Equation, que toutes les racines qui eſtoiont fauſſes deuienent vrayes, & par meſme moyen que toûtes celles qui eſtoiēt vrayes deuienent fauſſes : a ſçauoir en changeant tous les ſignes + ou -- qui ſont en la ſeconde, en la quatrieſme, en la ſixieſme, ou autres places qui ſe deſignent par les nombres pairs, ſans changer ceux de la premiere, de la troiſieſme, de la cinquieſme & ſemblables qui ſe deſignent par les nombres impairs.

Aaa 3

Post script

As Greenfield wrote in the extract quoted in answer to Qu. 2,

> it is to be lamented, that he did not exert his induſtry and ingenuity, rather to confirm than to deſtroy; rather to demonſtrate, how far we might rely on the method of negative quantities, than to overturn at once ſo great a part of the labours of the modern algebraiſts.

It is interesting to see how those who did try "rather to confirm than to destroy", went about the business. For example De Morgan*

> A father is fifty-six, and his son twenty-nine years old: when will the father be twice as old as the son? Let this happen x years from the present time; then the age of the father will be $56+x$, and that of the son $29+x$; and therefore, $56+x=2(29+x)=58+2x$, or $x=-2$. The result is absurd; nevertheless, if in the equation we change the sign of x throughout it becomes $56-x=58-2x$, or $x=2$. This equation is the one belonging to the problem: a father is 56 and his son 29 years old; when *was* the father twice as old as the son? the answer to which is, two years ago. In this case the negative sign arises from too great a limitation in the

---

* De Morgan, A., *On the study and difficulties of mathematics,* Open Court Pub. Co., 1898.

> terms of the problem, which should have demanded how many years have elapsed or will elapse before the father is twice as old as his son?
>
> Again, suppose the problem had been given in this last-mentioned way. In order to form an equation, it will be necessary either to suppose the event past or future. If of the two suppositions we choose the wrong one, this error will be pointed out by the negative form of the result. In this case the negative result will arise from a mistake in reducing the problem to an equation. In either case, however, the result may be interpreted, and a rational answer to the question may be given.

In this latter extract, De Morgan uses the same means as Frend, the change of sign in an equation, to change the sign of a root. But the treatment is qualitatively different.

We leave the final word to the philosopher Auguste Comte (1798-1857)*:

> NEGATIVE QUANTITIES.
>
> **As to negative quantities, which have given rise to so many misplaced discussions, as irrational as useless, we must distinguish between their *abstract* signification and their *concrete* interpretation, which have been almost always confounded up to the present day. Under the first** point of view, the theory of negative quantities can be established in a complete manner by a single algebraical consideration. The necessity of admitting such expressions is the same as for imaginary quantities, as above indicated; and their employment as an analytical artifice, to render the formulas more comprehensive, is a mechanism of calculation which cannot really give rise to any serious difficulty. ...

---

* Comte, A., *The Philosophy of Mathematics*. Translated by Gillespie, N.Y. 1851, p.81.

It is far from being so, however, with their concrete theory. This consists essentially in that admirable property of the signs + and −, of representing analytically the oppositions of directions of which certain magnitudes are susceptible. This *general theorem* on the relation of the concrete to the abstract in mathematics is one of the most beautiful discoveries ...

It consists in this: if, in any equation whatever, expressing the relation of certain quantities which are susceptible of opposition of directions, one or more of those quantities come to be reckoned in a direction contrary to that which belonged to them when the equation was first established, it will not be necessary to form directly a new equation for this second state of the phenomena; it will suffice to change, in the first equation, the sign of each of the quantities which shall have changed its direction; and the equation, thus modified, will always rigorously coincide with that which we would have arrived at in recommencing to investigate ...

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The Negative Numbers

Worksheet: George Peacock

In England, the death of Isaac Newton (1642-1727) was followed by a period of relatively low scientific activity in general, and in mathematics in particular. English mathematicians distanced themselves from other European mathematicians, in part because of the dispute between Newton and Leibniz (1646-1716) concerning precedence in the invention of the infinitesimal calculus. This chauvinism on the part of the English mathematicians, caused them to pay insufficient attention to mathematical advances on the continent. In addition, there were at that time men, like Frend and Masères, who wrote books against the use of negative numbers. The English mathematician George Peacock (1791-1858), together with others, began to work in order to "revive" English mathematics, and in particular, to put algebra on a sound scientific basis. To this end he wrote two books, *Arithmetical Algebra* (1842) and *Symbolical Algebra* (1845). In the distinction between these two types of algebra can be found Peacock's response to the arguments of Masères and Frend.*

Below we bring extracts from these books, in which Peacock explains the difference between arithmetical algebra and symbolical algebra and presents his *Principle of Permanence of Equivalent Forms,* by which he connects these two types of algebra.

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* About Peacock and his contribution to mathematics one can find further details in A. Macfarlane, *Lectures on Ten British Mathematicians of the Nineteenth Century,* Wiley, 1916.

In arithmetical algebra, we consider symbols as representing numbers, and the operations to which they are submitted as included in the same definitions (whether expressed or understood) as in common arithmetic: the signs + and − denote the operations of addition and subtraction in their ordinary meaning only, and those operations are considered as impossible in all cases where the symbols subjected to them possess values which would render them so...

...thus in... expressions... like $a - b$, we must suppose a greater than b...

all results whatsoever, including negative quantities, which are not strictly deducible as legitimate conclusions from the definitions of the several operations, must be rejected as impossible, or as foreign to the science. ...

Symbolical algebra adopts the rules of arithmetical algebra, but removes altogether their restrictions: thus symbolical subtraction differs from the same operation in arithmetical algebra in being possible for all relations of value of the symbols or expressions...

It is this adoption of the rules of the operations of arithmetical algebra as the rules for performing the operations which *bear the same names* in symbolical algebra, which secures the absolute identity of the results in the two sciences as far as they exist in common: or in other words, all the results of arithmetical algebra which are deduced by the application of its rules, and which are general in form, though particular in value, are results likewise of symbolical algebra, where they are general in value as well as in form: thus the product of a^m and a^n, which is a^{m+n} when m and n are whole numbers, and therefore general in form though particular in value, will be their product likewise when m and n are general in value as well as in form...

This principle, in my former Treatise on Algebra, I denominated the "*principle of the permanence of equivalent forms*," and it may be considered as merely expressing the general law of transition from the results of arithmetical to those of symbolical algebra...

Upon this view of the principles of symbolical algebra, it will follow that its operations are determined by the definitions of arithmetical algebra, as far as they proceed in common, and by the "principle of the permanence of equivalent forms" in all other cases...

The results therefore of symbolical algebra, which are not common to arithmetical algebra, are generalizations of form, and not necessary consequences of the definitions, which are totally inapplicable in such cases. It is quite true indeed that writers on algebra have not hitherto remarked the character of the transition from one class of results to the other, and have treated them both as equally consequences of the fundamental definitions of arithmetic or arithmetical algebra: and we are consequently presented with forms of demonstration, which though really applicable to specific values of the symbols only, are tacitly extended to all values whatsoever...

The definition of a *power*, in Arithmetical Algebra, implies that its index is a whole number: and if this condition be not fulfilled, the definition has no meaning, and therefore no conclusions are deducible from it: the principles, however, of Symbolical Algebra, will enable us, not merely to recognise the existence of such powers, but likewise to give, in many instances, a consistent interpretation of their meaning.

Questions

1. What is the difference between the expression $a - b$ in arithmetical algebra and in symbolical algebra?

2. What is the difference between what Peacock calls arithmetic and what we call arithmetic today?

3. In view of Peacock's definition of arithmetical algebra, symbolical algebra and the *Principle of Permanence of Equivalent Forms,* find where the principle is used in each of the following and if this use is justified.

   i) We know that

   $$a^m \cdot a^n = a^{m+n}, \qquad m, n \text{ natural numbers}.$$

   Substitute $n = 0$ to obtain

   $$a^m \cdot a^0 = a^{m+0} = a^m,$$

   Whence $a^m \cdot a^0 = a^m$.

   Therefore $a^0 = 1$.

   ii) We know that

   $$(a-b) \cdot (c-d) = a \cdot c - b \cdot c - a \cdot d + b \cdot d,$$

   when $c > d > 0$, $a > b > 0$.

   Substitute $a = 0$, $c = 0$, to obtain

   $$(-b) \cdot (-d) = +b \cdot d.$$

   iii) If a and b represent natural numbers, the distance of $a + b$ from 0 on the number line, is equal to the distance of a from 0 plus the distance of b from 0. We conclude that for every a, b the distance of $a + b$ from 0 is equal to the distance of a from 0 plus the distance of b from 0.

iv) If a, b and c are natural numbers and

$$a > b$$

then $\qquad ac > bc$.

We conclude that for every a, b and c

$$a > b \Rightarrow ac > bc .$$

v) For a, b natural numbers and $a > b$

$$(a^2 - b^2)/(a - b) = a + b .$$

We conclude that for every a and b

$$(a^2 - b^2)/(a - b) = a + b .$$

vi) For a, m natural numbers

$$a^m = \underbrace{a \cdot a \ \ldots \cdot a}_{m\ times}$$

Substitute $\qquad a = 0$,

whence $\qquad 0^m = 0$.

Substitute $\qquad m = 0$,

whence $\qquad 0^0 = 0$.

4. The following is an extract from the textbook *First Course in Algebra, Part I (SMSG, Yale Univ. Press, 1961, p. 145-6)*. Discuss the approach in the light of Peacock.

Chapter 7

PROPERTIES OF MULTIPLICATION

7-1. Multiplication of Real Numbers

Now let us decide how we should multiply two real numbers to obtain another real number. All that we can say at present is that we know how to multiply two non-negative numbers.

Of primary importance here, as in the definition of addition, is that we maintain the "structure" of the number system. We know that if a, b, c are any numbers of arithmetic, then

$$\begin{aligned} ab &= ba, \\ (ab)c &= a(bc), \\ a \cdot 1 &= a, \\ a \cdot 0 &= 0, \\ a(b + c) &= ab + ac. \end{aligned}$$

(What names did we give to these properties of multiplication?) Whatever meaning we give to the product of two real numbers, we must be sure that it agrees with the products which we already hav for non-negative real numbers and that the above properties of multiplication still hold for all real numbers.

Consider some possible products:

$(2)(3)$, $(3)(0)$, $(0)(0)$, $(-3)(0)$, $(3)(-2)$, $(-2)(-3)$.

(Do these include examples of every case of multiplication of pos tive and negative numbers and zero?) Notice that the first thre products involve only non-negative numbers and are therefore already determined:

$$(2)(3) = 6, \quad (3)(0) = 0, \quad (0)(0) = (0).$$

Now let us try to see what the remaining three products will have to be in order to preserve the basic properties of multiplication listed above. In the first place, if we want the multiplication property of 0 to hold for all real numbers, then we must have $(-3)(0) = 0$. The other two products can be obtained as follows:

$0 = (3)(0)$

$0 = (3)\big(2 + (-2)\big)$, by writing $0 = 2 + (-2)$; (Notice how this introduces a negative number into the discussion.)

$0 = (3)(2) + (3)(-2)$, if the distributive property is to hold for real numbers;

$0 = 6 + (3)(-2)$, since $(3)(2) = 6$.

We know from uniqueness of the additive inverse that the only real number which yields 0 when added to 6 is the number -6. Therefore, if the properties of numbers are expected to hold, the only possible value for $(3)(-2)$ which we can accept is -6.

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The Negative Numbers

Answer sheet: George Peacock

1. Peacock, in his *Symbolical Algebra,* explains it as follows:

> ... in the expression $a-b$, if we are authorized to assume a to be either *greater* or *less* than b, we may replace a by the equivalent expression $b+c$ in one case, and by $b-c$ in the other: in the first case, we get $a-b=b+c-b=b-b+c$ (Art. 22), $=0+c$ (Art. 16) $=+c=c$: and in the second $a-b=b-c-b$ $=b-b-c=0-c=-c$. The first result is recognized in Arithmetical Algebra (Art. 23): *but there is no result in Arithmetical Algebra which corresponds to the second:* inasmuch as it is assumed that no operation can be performed and therefore no result can be obtained, when a is less than b, in the expression $a-b$ (Art. 13).[*]
>
> 545. Symbols, preceded by the signs + or −, without any connection with other symbols, are called *positive* and *negative* (Art. 32) symbols, or *positive* and *negative* quantities: such symbols are also said to be *affected* with the signs + and −. *Positive* symbols and the numbers which they represent, form the subjects of the operations both of Arithmetical and Symbolical Algebra: but *negative* symbols, whatever be the nature of the quantities which the *unaffected* symbols represent, belong exclusively to the province of Symbolical Algebra.

---

[*] If we assume symbols to be capable of all values, from *zero* upwards, we may likewise include *zero* in their number: upon this assumption, the expressions $a+b$ and $a-b$ will become $0+b$ and $0-b$, or $+b$ and $-b$, or b and $-b$ respectively, when a becomes equal to *zero*: this is another mode of deriving the conclusion in the text.

2. Peacock, for example, does not regard the four operations with negative numbers as arithmetic, since the negative quantity does not have any place whatsoever in arithmetic, as we have seen in the previous question. And this is common to many mathematicians in the 19th century.

Nowadays, the word arithmetic has a much more wide meaning. We read in Lapede's *Dictionary of Physics and Mathematics* (McGraw-Hill, 1978) the following definition of arithmetic:

> *Addition, subtraction, multiplication and division usually of integers, rational numbers, real numbers, or complex numbers.*

Very often, operations between other mathematical entities (such as vectors, matrices, etc.) are also called arithmetic.

3. In each of the sections we have made use of the *Principle of Permanence of Equivalent Forms*. What we have to analyze in each case, is whether this use is legitimate or leads us to contradictions.

   i) When we substituted $n = 0$ in $a^m a^n = a^{m+n}$, we used the principle; that is, we extended the application of the equality from the domain of arithmetical algebra to that of symbolical algebra.
   In this case, its use is justified since it does not lead us to contradictions.

   ii) When we substitute $a = 0$ and $c = 0$, we use the principle.
   Again this is justified.

In these two sections, the principle does not lead us astray, but on the other hand, we should be explicitly aware of the "logical status" of what we have done.
In the following, we bring two extracts related to these two sections, which warn against regarding the arguments based on the principle as proofs. They are not proofs, unless we accept the principle as an axiom, which we

cannot do quite so easily since, as we shall see in subsequent sections, it can also lead to incorrect and contradictory conclusions.

The first extract is taken from S.I. Brown, Signed Numbers: a "Product" of misconceptions. *The Mathematics Teacher,* 1969, 62, p. 183-195.

Preservation principle

The preservation principle is that if we wish to extend a concept in mathematics beyond its original definition then that candidate ought to be chosen which leaves as many principles of the old system intact as possible. The fact that these principles are left intact does not at all *prove* that a definition for elements of an extended set is correct. It merely motivates us to make a particular definition.

...Suppose we define a^n for n a natural number to be

$$a \cdot a \cdot \ldots \cdot a,$$

where a appears n times as a factor. For m, n, belonging to the set of natural numbers, it is easy to prove that

$$a^m \cdot a^n = a^{m+n}.$$

Now what is a^0? It obviously has no meaning in terms of the definition of a^n, since it is meaningless to select the number a zero times as a factor. "Proofs" are often offered...

The difficulty with the above "proof" obviously lies with the assertion that

$$a^0 \cdot a^n = a^{0+n}.$$

Why should we assume that 0 behaves (as an exponent) the same way the natural numbers do? The answer is that we cannot, since the assertion

$$a^n \cdot a^m = a^{n+m}$$

was proven only for natural numbers. There would be nothing inconsistent about our system if we maintained the above equality only for natural numbers and stipulated that it failed in the case of $n = 0$. We could then define a^0 in any way we wished and still have a consistent system. If, however, we wish the principle for multiplication to be preserved in the *new* system (which includes 0), then we are forced to *define* a^0 to be 1 as the "proof" suggests.

The second extract comes from F. Klein, *Elementary Mathematics From an Advanced Standpoint. Arithmetic, Algebra, Analysis.* Dover Publications, n.d. p.23-28 (originally published in German in 1908.)

> ...If we now look critically at the way in which negative numbers are presented in the schools, we find frequently the error of trying to prove the logical necessity of the rule of signs, corresponding to the above noted efforts of the older mathematicians. One is to derive $(-b)(-d) = +bd$ heuristically, from the formula $(a - b)(c - d)$ and to think that one has a proof, completely ignoring the fact that the validity of this formula depends on the inequalities $a > b$, $c > d$[1]. Thus the proof is fraudulent, and the psychological consideration which would lead us to the rule by way of the principle of permanence is lost in favor of quasi-logical considerations.
>
> Of course the pupil, to whom it is thus presented for the first time, cannot possibly comprehend it, but in the end he must nevertheless believe it ...
>
> In opposition to this practice, I should like to urge you, in general, never to attempt to make impossible proofs appear valid. One should convince the pupil by simple examples, or, if possible, let him find out for himself that, in view of the actual situation, *precisely these conventions, suggested by the principle of permanence, are appropriate in that they yield a uniformly convenient algorithm, whereas every other convention would always compel the consideration of numerous special cases.* To be sure, one must not be precipitate, but must allow the pupil time for the revolution in his thinking which this knowledge will provoke. And while it is easy to understand that other conventions are not advantageous, one must emphasize to the pupil how really wonderful the fact is that a general useful convention really *exists*; it should become clear to him that this is by no means self-evident.

iii) This statement in mathematical form is

$$|a + b| = |a| + |b|$$

and this is true if a and b are natural numbers.

If we use the principle to extend the result to the non-negative numbers, we have made valid use of the principle, since it does not lead us to contradictions.

But if we use the principle to extend the result to negative numbers, we obtain a contradiction, as is clear from the following example,

$$|-3 + 5| = |-3| + |5|$$
$$2 = 3 + 5$$
$$2 = 8 .$$

iv) If we extend the result $a > b \Rightarrow ac > bc$, from the natural numbers to all the positive numbers, all is well. But, if we include zero, then $c = 0$ leads to

$$a > b \Rightarrow 0 > 0 ,$$

which contradicts the elementary meaning of >, and the concept of order which it is meant to express.

So, at least, we would require the exclusion of $c = 0$.

But worse is to come if we try to extend to the negative numbers. Suppose $a \neq 0$ and let $b = -a$, that is our result becomes

$$a > -a \Rightarrow ac > -ac$$

Now substitute $c = -1$ and we obtain

$$a > -a \Rightarrow a(-1) > -a(-1) ,$$

i.e. $a > -a \Rightarrow -a > a$.

which is completely contradictory to the basic property of order; that is, for any two numbers a, b, *just one* of the following three possibilities is correct,

$$a = b , \quad a > b , \quad a < b .$$

In conclusion, the *Principle of Permanence of Equivalent Forms* can be used to suggest the extension of concepts, but it is not a universal principle and one has to use it with care. If we look back, at previous worksheets, we can find hints of its implicit use.

- Saunderson notices that if we multiply the terms of an arithmetic series (of natural numbers) by a natural number, then the result will be also an arithmetic series. This property he extends to negative numbers and thus makes a justified use of the principle.

- Arnauld tries to use the principle on the ratio concept and fails, because in this case it is unjustified, as we saw in section iv).

- Bell in *The Development of Mathematics* discusses Peacock and his principle and sees in it a further weakness.

> It is difficult to see what the principle means, or what possible value it could have even as a heuristic guide. If taken at what appears to be its face value, it would seem to forbid $ab = -ba$, one of the most suggestive breaches of elementary mathematical etiquette ever imagined, as every student of physics knows from his vector analysis. As a parting tribute to the discredited principle of permanence, we note that since $2 \times 3 = 3 \times 2$, it follows at once from the principle that $\sqrt{2} \times \sqrt{3} = \sqrt{3} \times \sqrt{2}$. But the necessity for proving such simple statements as the last was one of the spurs that induced Dedekind in the 1870's to create his theory of the real number system. According to that peerless extender of the natural numbers, "Whatever is provable, should not be believed in science without proof."

Missing answers in Peacock answer sheet

v) For natural numbers (where the subtraction $a - b$ is possible if and only if $a > b$), we can assert that

$$\frac{a^2 - b^2}{a - b} = a + b \qquad (1)$$

since $a^2 - b^2 = (a + b)(a - b)$ and we may cancel the factor $a - b$.

If we extend the result from the natural numbers to the integers (and also to the real numbers), all is well except for the case when $a = b$.

If, for example, we substitute in (1) $a = b = \frac{1}{2}$, we have

$$\frac{0}{0} = 1$$

but if we substitute $a = b = 3$ we have

$$\frac{0}{0} = 6 .$$

In general, we have $\frac{0}{0} = 2a$. Thus the extension is invalid.

vi) In this case, we used Peacock's Principle twice. First, we extend a^m to include $a = 0$, whence $0^m = 0$. And all is well.

Then, we used the Principle again to extend to $m = 0$, obtaining $0^0 = 0$.
But another possible extension can be obtained from section i) of this question, where we found that $a^0 = 1$ for all a, also by means of the Principle - whence by a second extension, we have $0^0 = 1$.

So, we are in a similar situation to the previous section. In fact, both $\frac{0}{0}$, and 0^0 are indeterminate expressions.

And B. Russell* wrote:

> The possibility of a deductive Universal Algebra is often based upon a supposed principle of the Permanence of Form. Thus it is said, for example, that complex numbers must, in virtue of this principle, obey the same laws of addition and multiplication, as real numbers obey. But as a matter of fact there is no such principle. In Universal Algebra, our symbols of operation, such as + and ×, are variables, the hypothesis of any one Algebra being that these symbols obey certain prescribed rules. In order that such an Algebra should be important, it is necessary that there should be at least one instance in which the suggested rules of operation are verified. But even this restriction does not enable us to make any general formal statement as to all possible rules of operation. The principle of the Permanence of Form, therefore, must be regarded as simply a mistake: other operations than arithmetical addition may have some or all of its formal properties, but operations can easily be suggested which lack some or all of these properties.

From the philosophical point of view there is no reason why the principle should hold unless we want it to. But from the didactical point of view, as we saw in the previous sections, is often extremely useful.

4. The approach in the SMSG textbook could be considered as a conceptual inheritance from Peacock. In fact, the paragraph can be seen to be based on a critical use of Peacock's Principle. It is explicitly assumed that only a *specified* set of laws (commutative, associative, distributive, multiplication by zero) can be extended from the positive to the negative numbers and then become the axioms for the proofs which follow.

B. Russell, *The Principles of Mathematics*, Allen & Unwin Ltd., 1951, p. 377.

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The Negative Numbers

Worksheet: The formal entry of the negative numbers into mathematics

Part I: Definitions

In the long history of the negative numbers, there are periods in which advances were made, and periods of stagnation. But there are almost no "discoveries" on which one can put a name or a date. The first half of the nineteenth century saw a number of attempts to "legitimize" the various sets of numbers - negative numbers, real numbers and complex numbers - of which we have already seen one such attempt in the work of Peacock. Another attempt, which had practically no influence on subsequent work in this field, was that of Sir William Rowan Hamilton (1805-1865), in a paper entitled Theory of conjugate functions, or algebraic couples; with a preliminary and elementary essay on algebra as the science of pure time, published in 1837.* This paper "was an attempt to give a system of axioms or principles for analysis (for Hamilton, the Science of Algebra included analysis) so that it can be *a Science properly so called; strict, pure and independent; deduced by valid reasonings from its own intuitive principles; and thus not less an object of priori contemplation than Geometry...* . Hamilton believed that our intuitive notion of order in time is more deep-seated than that of order in space and, as geometry is founded on the latter, so can analysis be founded on the former. In his essay Hamilton attempts to use these principles to construct and explain positive and contra-positive (negative) numbers, fractional quantities (rational numbers), and the set of all real numbers, all towards showing that analysis, like geometry, can be a Science."* One of the reasons for Hamilton's lack of influence is"... besides a general dullness and heaviness of style, there is too much obscurity...", as he himself admits. But this paper contains many of the ideas used by other mathematicians later in the century, with more success. Hamilton's paper is too involved to use as a source in this story - and this is true of later sources of which we are aware. So, for the last chapter in our history of the negative numbers, we change the style of worksheet. As we have seen in previous worksheets, many problems arose as a result of the attempt to solve and explain the solution of equations (Frend, De Morgan, Descartes). In the spirit of history, we shall take the final step using a very simple equation, to construct the integers from the natural numbers.

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* In the Trans. Roy. Irish Academy, Vol. 17, 1837, p. 293-422.

At the end, we shall indicate how the complete set of negative numbers might be constructed from the natural numbers.

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* Mathews J., William Rowan Hamilton's Paper, 1837 on the Arithmetization of Analysis. *Archive for the History of the Exact Sciences,* 1978, Vol. 19, p. 177-200.

All that we have at our disposal to start with is the set of natural numbers, $\{1, 2, 3, \ldots\}$, and the operations on them which produce a natural number as outcome - i.e. the arithmetical algebra of Peacock.

Consider the equation

$$a + x = b , \qquad a, b \in N$$

where N is the set of natural numbers.

1. Under what conditions is it also true that $x \in N$? (In other words, when does the equation have a natural number solution?)

2. Given two equations with the *same* solution $x \in N$,

$$a + x = b$$

$$\text{and} \quad a_1 + x = b_1,$$

what is the relation between a, b and a_1, b_1 ?

3. The solution of $a + x = b$, if it exists, is determined uniquely by a and b. Hence, we may denote the solution by the ordered pair (a, b).

   i) To which natural number does the pair (a, b) correspond?
   ii) Find *all* the ordered pairs corresponding to the natural number n. All these ordered pairs are said to be *equivalent*.

4. Draw the graph of all the pairs corresponding to the natural numbers 1, 2, etc.

5. We know how to add natural numbers - and we have seen how we may represent a natural number by an ordered pair. Hence we can "add" the ordered pairs corresponding to natural numbers.

i) Define the addition of such pairs, and examine it for validity.

ii) Can you "add" ordered pairs without restriction, i.e. without leaving N and its arithmetic?

iii) Give the addition of pairs a graphical interpretation.

6. i) Define the "subtraction" of ordered pairs (which represent national numbers) and examine it for validity.

ii) Can you "subtract" ordered pairs at will?

iii) Give subtraction of pairs a graphical representation.

7. Consider again the original equation with a natural number solution

$$a + x = b , \qquad a, b \in N .$$

Write it in terms of ordered pairs.

8. In section 3 we saw that not all ordered pairs correspond to natural numbers. Which do not?

Now let's generalise. Let (a, b) be *any* ordered pair of natural numbers. Generalise the concept of *equivalent* pairs using *addition* of natural numbers only.

Why do you think we suggested using addition only, and not subtraction?

9. i) Return to sections 4 - 7 and generalise them in the light of the generalisation in section 8.

ii) In what sense is it now true to say that every equation of the form

$$a + x = b \,, \quad a, b \in N \,,$$

has a solution?

10. The extension achieved in sections 8 and 9 might be regarded as a "didactical heritage" of Peacock's law of permanence of equivalent forms.

    In what sense is it similar, and in what *different* ?

11. In our new system of all ordered pairs of natural numbers, identify the set of equivalent pairs which from now on we shall denote by

    $$n \,,\, 0 \,,\, -n \qquad n \in N$$

    respectively.

    A set of equivalent pairs we shall call an *integer*, and the set of integers is denoted by Z.

12. Examine the following statements.

    i) For all $z_1 , z_2 \in Z$,

    $$z_1 + z_2 \in Z$$

    and $$z_1 - z_2 \in Z \,.$$

    ii) Examine the following statement

    For all $z_1, z_2 \in Z$

    $$z_1 + x = z_2$$

    has a solution $x \in Z$.

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The Negative Numbers

Answer Sheet: The formal entry of the negative numbers into mathematics

Part I: Definitions

1. The equation $a + x = b$, $a, b \in N$, has a solution belonging to N if and only if $b > a$.

2. The two equations

$$a + x = b$$

$$a_1 + x = b_1$$

have the same solution $x \in N$, if and only if

$$b - a = b_1 - a_1 \quad \text{and} \quad b > a \text{ , } b_1 > a_1 .$$

The first half of this condition is equivalent to

$$b + a_1 = b_1 + a ,$$

to which we shall have cause to refer later.

3. i) The ordered pair (a, b) corresponds to the natural number $b - a$.

   ii) All the ordered pairs corresponding to $n \in N$ satisfy

$$b - a = n \qquad b > a \text{ , } a, b \in N .$$

That is $\quad b = a + n$.

Hence all the pairs of the form $(a, a + n)$, $a \in N$ correspond to n.

For example, (8, 10), (2, 4), (100, 102) etc., all correspond to the number 2. All the pairs which represent a given natural number, are called *equivalent* pairs, and we denote this by the symbol $\sim$, for example:

$$(2, 4) \sim (8, 10).$$

Three Methodological Notes

I. A *number* in mathematics is an abstract concept which is represented in many ways. The simplest of these is the single *numeral*; for example 3 and III are numeral representations of the number concept three. Another representation is 11_2 (in base 2). In the process of doing mathematics, we represent numbers in many forms - thus $5 + 3 - 6 + 1$ is also a representation of the number 3. In the context in which we are working at the moment, we are looking at certain particular representations of the natural numbers. The first is the representation in the form $b - a$, $b, a \in N$. This form is the natural one in the context, but it has one deficiency, the condition $b > a$, without which the expression $b - a$ does not make any sense. Therefore, we replaced this by another form of representation (a, b), for which it is still true that it represents a natural number if and only if $b > a$, but the ordered pair is still a legitimate symbol without this condition, since we have got rid of the operation which is undefined for $b < a$. The advantage of this apparently trivial technical step is that we now have a notation which generalises without difficulty, and as we see in the following, we thus obtain the integers. It should be remembered that all we are doing throughout is formalising intuition, and although the use of $b - a$,

$b < a$, is formally illegitimate, intuitively it is entirely legitimate and almost essential as a guide to the formal steps.

II. There is another point, normally intuitively implicit in much practical mathematical work, which we need to make explicit in our present formal context. We first illustrate it by familiar example and then return to our context.

$$\frac{1}{2}, \frac{2}{4}, \frac{35}{70}, \frac{251}{502}, \ldots$$

are all representations of the same rational number. (There are, of course, many other representations of other forms, e.g. 0.5, but we are considering the one form only.) Sometimes a distinction is made between a *fraction,* which is one of these representations, and the *rational number* which is in some way the whole set of representations - the concept.

When we write

$$\frac{2}{4} = \frac{4}{8}$$

we are saying that $\frac{2}{4}$ and $\frac{4}{8}$ are two representations of the same number. The "fraction in its lower terms" is one way of choosing a particular representative from the set of representations. Given any two fractions, $\frac{a}{b}$ and $\frac{c}{d}$, they represent the same number if and only if

$$ad = bc .$$

Now let us rewrite this in formal mathematical language. Consider the set of all positive fractions $\frac{a}{b} : a, b \in N$. We say that two fractions $\frac{a}{b}$, $\frac{c}{d}$ are *equivalent* if and only if

$$ad = bc\ .$$

A subset (class) of equivalent fractions is called a rational number. Thus a rational number is a set of equivalent fractions. It is clear that we can remove the fraction from the development, and base our definition on the natural numbers only. All we have to do is to replace $\frac{a}{b}$ by the ordered pair (a, b). The condition for equivalence of such pairs remains the same since it is expressed entirely within N and operations on elements of N.

If we now define the arithmetic operations on the pairs (a, b), taking care to use definitions entirely valid within N, we have created the arithmetic of the positive rational numbers by formal mathematical definition from the arithmetic of the natural numbers.

This is precisely the logical skeleton of the method we are using to develop the arithmetic of the integers from the natural numbers.

The equation $a + x = b$ and the expression $b - a$ are the intuitive and motivational props; the ordered pair (a, b) is the basic element in our logical structure. The main point in the first three questions of this worksheet is the fact, that corresponding to any natural number, there are many representations in the form $b - a$, and hence in the form (a, b). So we introduce the idea of the set of all representations of a natural number, i.e. the idea of equivalent representations:

$$(a, b) \sim (c, d) \text{ if and only if } b - a = d - c$$

However there is a technical problem with this condition, which is the necessity for $b > a$, and it is precisely this condition which we wish to relax in order to build the set of integers. So we recast the equivalence in the form

$$(a, b) \sim (c, d) \text{ if and only if } b + c = a + d$$

which makes sense for all $a, b, c, d \in N$.
Now we are ready to use our intuition and to start defining the arithmetic of these ordered pairs and thus obtain the integers from the natural numbers on a rigorous mathematical basis.

III. Involved in the above argument is the concept of equivalence, which is a general mathematical concept of some importance. It is the mathematical formalisation of the physical sorting process. A proper unambiguous sorting of a set of objects requires that

i) every object is sorted;
ii) the categories into which the objects are sorted is not dependent on the order of sorting;
iii) an object is assigned to one and only one category.

Thus a collection of coloured plastic geometric shapes can be properly sorted according to shape or colour (or both), according to area, etc., where by the latter we mean that two shapes with the same area belong to the same category. On the other hand, if we instruct someone to sort the shapes placing in the same category any shape which has area greater than or equal to the shapes already there, he will fail to sort the shapes either altogether, or at least uniquely.

The mathematical formulation of the sorting process can be expressed as follows. Given a set S of elements a, b, c, ..., then a relation (denoted by $\sim$) between the elements of the set is an *equivalence relation* if for all, a, b and c $\in$ S

i) $a \sim a$, (reflexive property)

ii) $a \sim b \implies b \sim a$, (symmetric property)

iii) $a \sim b,\ b \sim c \implies a \sim c$, (transitive property).

(Compare with the three conditions of sorting above.) It can be proved that if these conditions are satisfied, then S can be sorted by $\sim$ into distinct *equivalence classes*.

In the above discussion we gave the example of fractions. Two fractions $\frac{m_1}{m_2}$ and $\frac{n_1}{n_2}$ are equivalent if and only if $m_1n_2 = m_2n_1$. Show that this relation on the set of fractions satisfies the three properties.

To return to our context: two ordered pairs of natural numbers (a, b) and (c, d), $b > a$ and $d > c$ are equivalent if and only if

$$\left.\begin{aligned} b &= a + n \\ d &= c + n \end{aligned}\right\} \text{ for some } n \in N,$$

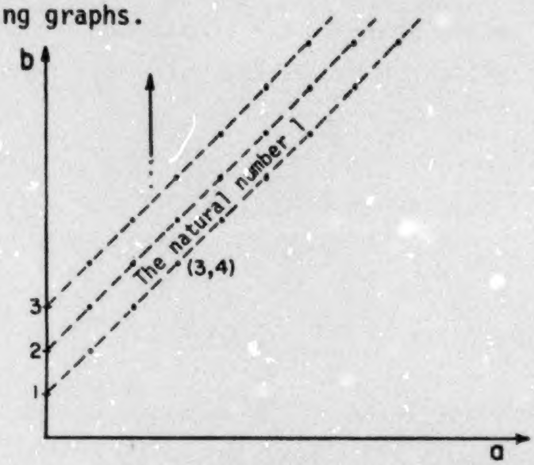
$$\text{or} \quad a + d = b + c.$$

Show that this relation on the set of ordered pairs satisfies the three properties.

Note that we can drop the conditions $b > a$ and so obtain an equivalence relation on the set of *all* ordered pairs (a, b), $a, b \in N$. See answer to Qu. 7.

Continuation of solutions

4. According to the choice of axes, we obtain one of the two following graphs.



Note that an equivalence class is represented by points all lying on the same straight line. The various lines do not intersect. To what general property of equivalence relations does this correspond?

5. i) $(a, b) \longrightarrow b - a \in N$

$(c, d) \longrightarrow d - c \in N$

Hence $(a, b) + (c, d) \longrightarrow b - a + d - c$

$= b + d - (a + c) \in N$

Thus we *define* $(a, b) + (c, d) = (a + c, b + d)$
The validity of this definition depends on two things - the first is that our definition is entirely within the arithmetic of N, which is

obviously the case - the second is that it is independent of the pairs we chose to represent the natural numbers. That is, if we choose any other representatives, say $(a_1, b_1) \sim (a, b)$ and $(c_1, d_1) \sim (c, d)$ then $(a_1 + c_1, b_1 + d_1) \sim (a + c, b + d)$.

The verification of this is a simple technical exercise. By the definition of equivalent pairs we have

$$b_1 - a_1 = b - a$$

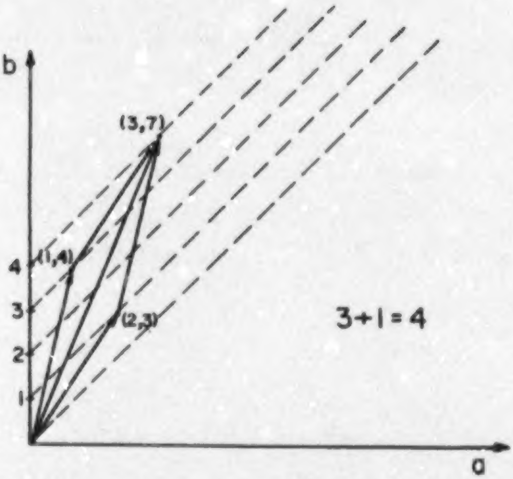
$$d_1 - c_1 = d - c .$$

Hence $\quad b_1 + d_1 - (a_1 + c_1) = b + d - (a + c)$

i.e. $\quad (a_1 + c_1, b_1 + d_1) \sim (a + c, b + d)$.

ii) We have no restriction on the addition of ordered pairs, since $a + c$ and $b + d$ always belong to N.

iii) In order to perform the addition on the graph, we choose any two representatives of the numbers to be added, i.e. two points on the lines which represent the numbers. We draw the arrows from the origin to these two points to obtain two adjacent sides of a parallelogram, which we complete by drawing the other two sides. The fourth vertex of the parallelogram so obtained is a point on the line representing the sum of the two numbers. In the figure we have represented the addition $3 + 1 = 4$.



Note that if we were to take any two other representatives (points) of the same number, the result would be the same - i.e. we would obtain another representative (point) of the same number.

A Fourth Methodological Note

IV. Let S be a set on which an equivalence relation $\sim$ has been defined, and let $\circ$ be a closed binary operation on S. (A closed binary operation is a rule for combining any two elements of S to obtain a unique element of S; e.g. addition on N, but subtraction, although a binary operation is not closed on N.) The equivalence relation $\sim$ defines equivalence classes: thus if $a \in S$ we denote by $\tilde{a}$ the equivalence class to which a belongs, and by $\tilde{S}$ the set of equivalence classes. It now seems reasonable that corresponding to $\circ$ on S, we should define $\tilde{\circ}$ on $\tilde{S}$ by

$$\tilde{a}\,\tilde{\circ}\,\tilde{b} = \widetilde{a \circ b}\ .$$

(It is usual to denote $\tilde{\circ}$ also by $\circ$, once one has got used to the idea.)

That is to combine two equivalence classes, we take a representative from each, combine the representatives, and identify the class to which the result belongs.

The procedure is described by the following diagram

$$\begin{tikzcd}
\tilde{a}, \tilde{b} \arrow[r, "①"'] \arrow[d, dashed] & a , b \arrow[d, "②"'] \\
\widetilde{a \circ b} & a \circ b \arrow[l, "③"']
\end{tikzcd}$$

The only problem is in the first step, since it is not unique. If $a_1 \varepsilon \tilde{a}$ and $b_1 \varepsilon \tilde{b}$, we would just as well have obtained

$$\tilde{a} \,\tilde{\circ}\, \tilde{b} = \widetilde{a_1 \circ b_1}$$

and there is no a priori reason to suppose that

$$\widetilde{a_1 \circ b_1} = \widetilde{a \circ b} \, .$$

It is precisely this that we checked at the end of the first part of section 5 above. The point can be made clear by returning to our example of fractions and rational numbers.

In the set of fractions we can define all sorts of binary operations. For example, consider the following two

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd} \qquad \frac{a}{b} \circ \frac{c}{d} = \frac{a + c}{b + d} \; .$$

It is a technical exercise to prove that the first, the usual addition of fractions, can be transferred to the addition of rational numbers, since any equivalent fractions will produce an equivalent result. The second, however, which is a common student error for addition, *cannot* be transferred to give the combination of rational numbers, as the following numerical examples show.

$$\begin{tikzcd}
\overset{\sim}{\frac{1}{2}}, \overset{\sim}{\frac{1}{4}} \arrow[r, "\textcircled{1}"'] \arrow[d, dashed] & \frac{1}{2}, \frac{1}{4} \arrow[d, "\textcircled{2}"] \\
\overset{\sim}{\frac{1}{3}} & \frac{1+1}{2+4} = \frac{2}{6} \arrow[l, "\textcircled{3}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\overset{\sim}{\frac{1}{2}}, \overset{\sim}{\frac{1}{4}} \arrow[r, "\textcircled{1}"'] \arrow[d, dashed] & \frac{2}{4}, \frac{1}{4} \arrow[d, "\textcircled{2}"] \\
\overset{\sim}{\frac{3}{8}} & \frac{2+1}{4+4} = \frac{3}{8} \arrow[l, "\textcircled{3}"']
\end{tikzcd}$$

and $\overset{\sim}{\frac{1}{3}} \neq \overset{\sim}{\frac{3}{8}}$, so the combination of $\overset{\sim}{\frac{1}{2}}$ and $\overset{\sim}{\frac{1}{4}}$ is not uniquely defined.

Thus whenever we define a binary operation between classes of objects by choosing representatives from the classes and combining these, we have to be careful that the definition is independent of the choice of representative.

In our context we wish to combine natural numbers as represented by classes of ordered pairs, by the definition

$$(a, b) + (c, d) = (a + c, b + d)\,.$$

But there are many pairs equivalent to (a, b) and (c, d), and we have to check that the combination of any equivalent pair always results in a pair equivalent to (a + c, b + d) .

Continuation of solutions

6. i) and ii) The natural definition of subtraction would seem to be

$$(a, b) - (c, d) = (a - c, b - d)\,.$$

But if we choose (2, 5) corresponding to 3 and (8, 9) corresponding to 1, then although 3 - 1 is possible, by our definition we have

$$(2, 5) - (8, 9) = (2 - 8, 5 - 9)$$

which is illegitimate.

So we start again.

(a, b) represents $b - a$ and (c, d) represents $d - c$, and we can perform the subtraction in N if and only if

$$b - a > d - c ,$$

to obtain $b - a - (d - c) = (b + c) - (a + d)$,

which belongs to N and can be represented by $(a + d, b + c)$

Hence we define

$$(a, b) - (c, d) = (a + d, b + c) .$$

Now we don't have any subtraction on the right-hand side, so the former trouble has disappeared. But note that the pair $(a + d, b + c)$ represents a natural number if and only if

$$b + c > a + d$$

$$\Longleftrightarrow b - a > d - c ,$$

which was our initial condition above - so all is well. It still remains to prove that the definition is valid, i.e. that it is independent of the pairs chosen to represent the natural numbers. This is entirely similar to the proof in section 5.

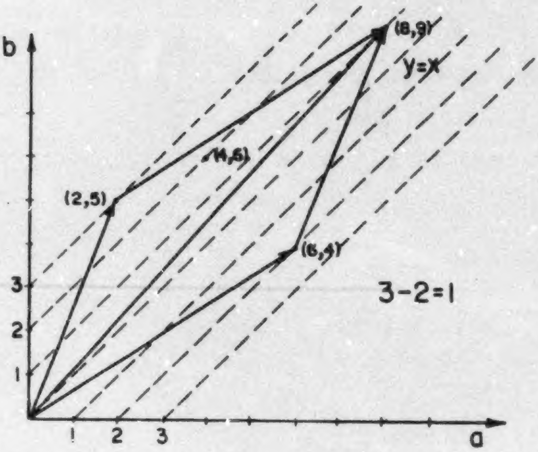
iii) Our definition of subtraction is equivalent to

$$(a, b) - (c, d) = (a, b) + (d, c)$$

(although (d, c) does not represent a natural number, since d > c).

Hence we can draw the graphical representation as for addition, except that we shall make use of points "below" the line y = x, which do not represent natural numbers. (Note that (d, c) is obtained by reflecting (c, d) in the line y = x.)

As we shall see later if (a, b) corresponds to n ε N, then (b, a) corresponds to -n.



7. In this question we only change representation, but otherwise all remains unchanged. The "new" equation can be written

$$(1,\ 1 + a) + (1,\ 1 + x) = (1,\ 1 + b)$$

or some equivalent form. The solution is

$$(1,\ 1 + x) = (1,\ 1 + b) - (1,\ 1 + a) = (2 + a,\ 2 + b)$$

which is equivalent to (a, b).

8. If $a > b$, then (a, b) does not represent a natural number. The definition of equivalence as given in sections 2 and 3 was

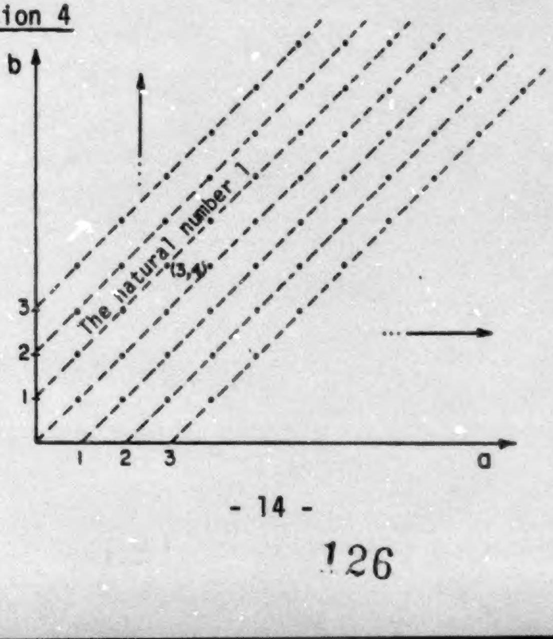
$(a, b) \sim (c, d)$ if and only if $a + d = b + c$.

This definition makes sense even if we remove the restriction $b > a$, because we remain within the set N and its arithmetic.
This is the reason we suggested using addition only - because if we had used the alternative form

$(a, b) \sim (c, d)$ if and only if $b - a = d - c$

we would not be able to remove the condition $b > a$.

9. i) Section 4



The graph now extends on both sides of the line $y = x$.

Section 5. Addition can be defined in exactly the same way as before, even though we are now "adding" pairs that do not correspond to natural numbers.

Section 6. Subtraction is also defined the same way, only we remove the condition $b + c > d + a$.

Section 7. We drop the condition $b > a$ and hence obtain the rewritten equation in the form

$$(1,\ 1 + a) + (x_1,\ x_2) = (1,\ 1 + b)$$

Note that we have to write (x_1, x_2) and cannot write $(1, 1 + x)$ as before, because the solution may not now belong to N.

The formal solution of this equation becomes

$$\begin{aligned}(x_1,\ x_2) &= (1,\ 1 + b) - (1,\ 1 + c)\\ &= (2 + a,\ 2 + b) \text{ by the extension of section 6,}\end{aligned}$$

which is equivalent to (a, b) as before.

ii) Formally we have answered this question in the immediately preceding note. But we have to realise that the solution (a, b) involves a conceptually new entity when $b < a$. See also section 12 ii).

10. The extension in section 8 is very similar to that of Peacock's extension by the principle of permanence of equivalent forms. The difference lies in the fact that we have been very particular to define our extension in terms of the known arithmetic in N, whereas Peacock's principle leads to expressions like $b - a$, which are

essentially meaningless, and are to be manipulated by known rules, also not too carefully specified. As we have seen, this lack of precision can lead us to trouble. Whereas Peacock extends by leaps and bounds our extension is painstaking brick upon brick.

11. All the equivalent ordered pairs of the form $(k, n + k)$, $k \in N$ are denoted by n, as before.
(Intuitively, 0 corresponds to $a + x = a$, i.e. (a, a).)
So we denote the set of equivalent ordered pairs of the form (k, k), $k \in N$, by 0.
(By definition, $(a, b) + (b, a) = (a + b, a + b)$, which latter we denoted by 0. Hence if (a, b) is denoted by n, (b, a) is denoted by $- n$.)
So we denote the set of equivalent ordered pairs of the form $(n + k, k)$, $k \in N$ by $-n$.
Note that because our definitions of addition and subtraction are independent of the ordered pair chosen to form a set of equivalent pairs, we can add and subtract integers.
Note also that in the graphical representation, the line representing n is symmetric, with respect to the line representing 0, to the line representing $-n$.

12. i) Since the addition and subtraction of ordered pairs always gives another ordered pair, and Z is the set of equivalence classes of all ordered pairs, both $z_1 + z_2$ and $z_1 - z_2$ belong to Z for all $z_1, z_2 \in Z$.

Hence, unlike, N, Z is also closed for subtraction.

ii) The temptation is to write

$$z_1 + x = z_2 \implies x = z_2 - z_1 \in Z \text{ by part i)}$$

But we have only defined addition and subtraction, we have not proved any of their properties - e.g. that addition is commutative and associative. In fact, the proofs of all these properties are simple technical manipulations. For example, we require the property

$$0 + z = z \qquad \text{for all } z \in Z$$

Proof Represent 0 and z by ordered pairs, i.e.

$$0 + z \text{ is represented by } (a, a) + (c, d), \quad a, c, d \in N$$

Now, by definition $(a, a) + (c, d) = (a + c, a + d)$ and, by the definition of equivalence, the latter is equivalent to (c, d).

If we write out the solution of $z_1 + x = z_2$ in full, it looks like this

$$\begin{aligned}
& & z_1 + x &= z_2 & & \\
&\implies & -z_1 + (z_1 + x) &= -z_1 + z_2 & & \\
&\implies & (-z_1 + z_1) + x &= -z_1 + z_2 & & \text{associativity of } + \\
&\implies & 0 + x &= -z_1 + z_2 & & -z_1 + z_1 = 0 \\
&\implies & x &= -z_1 + z_2 & & 0 + x = x \\
&\implies & x &= z_2 + (-z_1) & & \text{commutativity of } + \\
& & x &= z_2 - z_1 & & \text{addition of } -z_1 \text{ is equivalent to subtraction of } z_1.
\end{aligned}$$

We content ourselves with the knowledge that each of these properties can be proved, but this should not be taken as asserting that all the properties of arithmetic in N can be transferred to Z. For example, we have already seen that subtraction is not closed in N, but is closed in Z. And when we come to multiplication there will be properties in N which do not extend to Z.

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The Negative Numbers

Worksheet: The formal entry of the negative numbers into mathematics

Part II: Proofs

In this sheet we continue the development of the formal definition of negative numbers. We shall formally prove some of the properties that constituted "stumbling blocks" in the history of negative numbers, precisely because of the absence of mathematical definitions.

Since Part I has been full of technical steps, our first question will review what we have done, summarizing it.

1. Recalling that we use

⇢ to denote "corresponds to"

∿ to denote "equivalent to"

⇒ to denote "implies"

⇔ to denote "if and only if".

complete the following synopsis:

Sections 1 - 3

Ordered pairs $a, b, c, d, \in N$

$$\begin{tikzcd} a + x = b\ ,\ x \in N \arrow[r, Rightarrow] \arrow[d] & \underline{\quad > \quad} \\ (\ ,\ ) \arrow[r] & \underline{\qquad} \in N \end{tikzcd}$$

$$(a, b) \sim (c, d) \Leftrightarrow \underline{\quad + \quad = \quad + \quad}$$

Sections 5 - 6

Addition and subtraction $n, m, a, b \in N$

$$\begin{tikzcd} & n + m \arrow[dl] & \\ (a, a + n) + (b, b + m) \arrow[ur] & \overset{=}{\text{by def}} & \underline{(\qquad , \qquad)} \end{tikzcd}$$

$$\begin{tikzcd} & n - m\ ,\ n > m \arrow[dl] & \\ (a, a + n) - (b, b + m) \arrow[ur] & \overset{=}{\text{by def}} & \underline{(\qquad , \qquad)} \end{tikzcd}$$

Section 11

Integers

$a, b \in N \quad b > a$

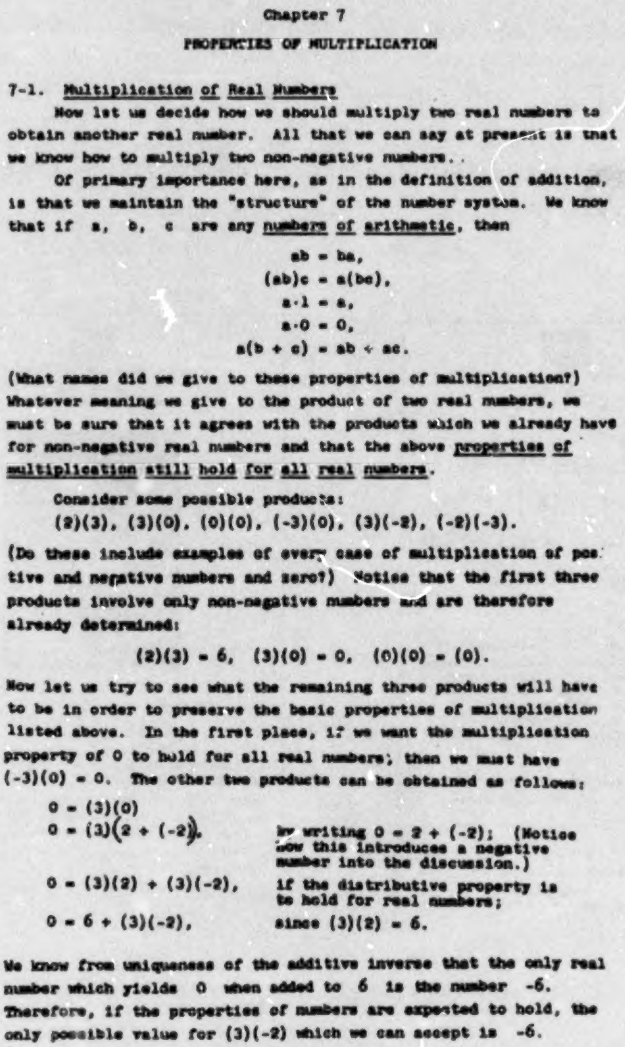
$n \longrightarrow (a, b)$

$0 \longrightarrow \underline{(\ ,\ )}$

$-n \longrightarrow \underline{(\ ,\ )}$

2. Define multiplication in Z, and examine it for validity. (Use the relevant ideas from Part I : ordered pairs, equivalence, etc.)

3. In the Peacock worksheet an extract of an SMSG textbook was to be discussed in the light of Peacock's Principle.



Chapter 7

PROPERTIES OF MULTIPLICATION

7-1. Multiplication of Real Numbers

Now let us decide how we should multiply two real numbers to obtain another real number. All that we can say at present is that we know how to multiply two non-negative numbers.

Of primary importance here, as in the definition of addition, is that we maintain the "structure" of the number system. We know that if a, b, c are any numbers of arithmetic, then

$$ab = ba,$$
$$(ab)c = a(bc),$$
$$a \cdot 1 = a,$$
$$a \cdot 0 = 0,$$
$$a(b + c) = ab + ac.$$

(What names did we give to these properties of multiplication?) Whatever meaning we give to the product of two real numbers, we must be sure that it agrees with the products which we already have for non-negative real numbers and that the above properties of multiplication still hold for all real numbers.

Consider some possible products:
$(2)(3)$, $(3)(0)$, $(0)(0)$, $(-3)(0)$, $(3)(-2)$, $(-2)(-3)$.

(Do these include examples of every case of multiplication of positive and negative numbers and zero?) Notice that the first three products involve only non-negative numbers and are therefore already determined:

$$(2)(3) = 6, \quad (3)(0) = 0, \quad (0)(0) = (0).$$

Now let us try to see what the remaining three products will have to be in order to preserve the basic properties of multiplication listed above. In the first place, if we want the multiplication property of 0 to hold for all real numbers, then we must have $(-3)(0) = 0$. The other two products can be obtained as follows:

$0 = (3)(0)$

$0 = (3)(2 + (-2))$, by writing $0 = 2 + (-2)$; (Notice how this introduces a negative number into the discussion.)

$0 = (3)(2) + (3)(-2)$, if the distributive property is to hold for real numbers;

$0 = 6 + (3)(-2)$, since $(3)(2) = 6$.

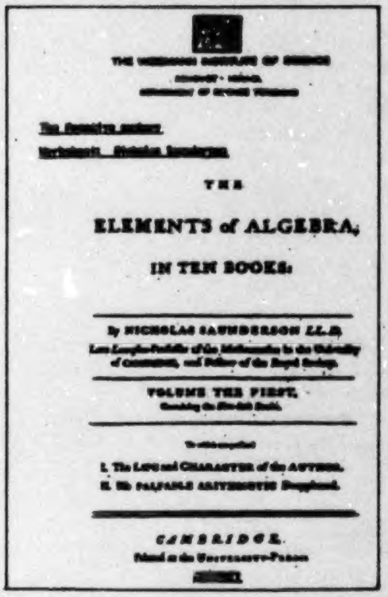
We know from uniqueness of the additive inverse that the only real number which yields 0 when added to 6 is the number -6. Therefore, if the properties of numbers are expected to hold, the only possible value for $(3)(-2)$ which we can accept is -6.

The multiplication property of zero and the distributive law are extended from the positive to the negative numbers in order to conclude the law of signs of multiplication.

In the formal development given here we do not have to make these extensions, we can prove them.

i) Prove that $a \cdot 0 = 0$ for all $a \in Z$

ii) Prove that multiplication is distributive over addition.



THE

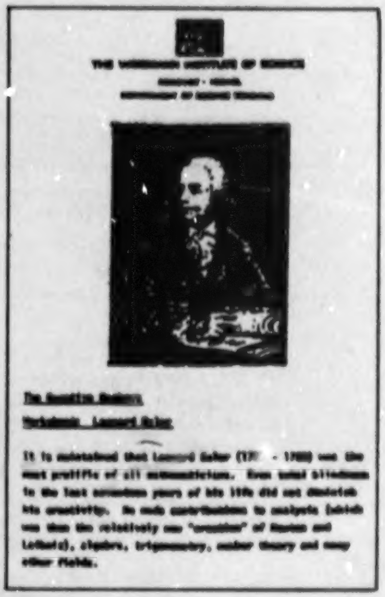
ELEMENTS of ALGEBRA,

IN TEN BOOKS:

By NICHOLAS SAUNDERSON LL.D.

VOLUME THE FIRST,

CAMBRIDGE.



4. In the Saunderson worksheet and in the Euler worksheet

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we saw ways of introducing the multiplication of negative numbers; in particular, the case of two negative factors. The excerpts brought in those sheets were discussed both from the didactical and the mathematical point of view. Didactically we compared the strategies among themselves, and also with the presentations in current textbooks. Mathematically we pointed out some "weaknesses" of the presentations, principally due to the lack of mathematical definition of negative numbers. These weaknesses made it difficult to answer the rejectionists such as Frend and Maseres. Now, at this stage of development of the formal mathematics we are able *to prove*, for instance, that the product of two negative numbers is a positive number.

Prove the "law of signs" of multiplication.

5. In the Euler worksheet we saw how the author may have tacitly used the assumption that the cancellation law can be extended from the positive to the negative numbers.

   Now, again, we can *prove* that this law holds.

   Prove that for $a \neq 0$, $ab = ac \Rightarrow b = c$ for all $b, c \in Z$.

Part III : And so on ...

1. For all $a, b \in Z$ does $ax = b$ have a solution in Z ?

   What do you suggest ? ...

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The Negative Numbers

Answer sheet: The formal entry of the negative numbers into mathematics

Part II: Proofs

1. Ordered pairs $a, b, c, d \in N$

$$\begin{tikzcd} a + x = b,\ x \in N \arrow[d] & \Rightarrow & \underline{b > a} \\ \underline{(a, b)} \arrow[rr] & & \underline{b - a \in N} \end{tikzcd}$$

$$(a, b) \sim (c, d) \Leftrightarrow \underline{a + d = b + c}$$

Addition and subtraction $n, m, a, b \in N$

$$n + m$$

$$\diagup \qquad \diagdown$$

$$(a, a + n) + (b, b + m) \underset{\text{by def}}{=} \underline{(a + b, a + n + b + m)}$$

$$n - m \qquad n > m$$

$$\diagup \qquad \diagdown$$

$$(a, a + n) - (b, b + m) \underset{\text{by def}}{=} \underline{(a + b + m, a + n + b)}$$

Integers $\quad a, b \in N \quad b > a$

$$n \rightarrow (a, b)$$

$$0 \rightarrow \underline{(a, a)}$$

$$-n \rightarrow \underline{(b, a)}$$

2. In order to obtain a definition of multiplication in Z, we go back and look at what it means for those ordered pairs which correspond to natural numbers, where multiplication is already defined.

$$(a, b) \rightarrow b - a \in N$$

$$(c, d) \rightarrow d - c \in N$$

Hence

$$\begin{aligned}(a, b)\cdot(c, d) \rightarrow (b - a)\cdot(d - c) &= bd - ad - bc + ac \\ &= bd + ac - (ad + bc) \in N\end{aligned}$$

Thus we *define*

$$(a, b) \cdot (c, d) = (ad + bc, bd + ac)$$

Here again, the validity of this definition depends on two things. The first is that our definition is entirely within the arithmetic of N, which is obviously the case. The second is that it is independent of the pairs we choose to represent the natural numbers:

that is, if we take other pairs, equivalent to those we have here, to show that the result of their multiplication is also an equivalent pair. This is again a technical exercise, which does not make very instructive reading - so we just give the outline of the proof.

Suppose $(a, b) \sim (a_1, b_1)$, i.e. $a + b_1 = a_1 + b$ (1)

and $(c, d) \sim (c_1, d_1)$, i.e. $c + d_1 = c_1 + d$ (2)

we have to show that

$$(a, b)\cdot(c, d) \sim (a_1, b_1)\cdot(c_1, d_1)$$

i.e. $(bc + ad, ac + bd) \sim (b_1c_1 + a_1d_1, a_1c_1 + b_1d_1)$

i.e. $bc + ad + a_1c_1 + b_1d_1 = ac + bd + b_1c_1 + a_1d_1$ (3)

It remains to derive (3) from (1) and (2).

Now, if we drop the constraint $b > a$, $d > c$, we still remain within the arithmetic of N when we define

$$(a, b) \cdot (c, d) = (ad + bc, bd + ac)$$

Thus the definition of multiplication of integers rests solely on addition and multiplication in N.

3. i) By definition, if a is represented by (a_1, a_2) and 0 by by (m, m),

$$(a_1, a_2)\cdot(m, m) = (a_1m + a_2m, a_1m + a_2m)$$

and the latter is just another pair which represents 0.

ii) One has to prove that

$$(a, b)\cdot[(c, d) + (e, f)] = (a, b)\cdot(c, d) + (a, b)\cdot(e, f),$$

which is only a matter of patience.

Note that since we are working with ordered pairs, rather than sets of equivalent ordered pairs, the = sign could be replaced by the $\sim$ sign, which is less restrictive. But this is not necessary here.

4. We saw that pairs of the form $(k, k + n)$ represent the positive number n, for all $k \varepsilon N$, and of the form $(k + n, k)$ the negative number $-n$.

   Thus if, for example, we wish to prove that "negative x negative = positive", we have to show that

   $$(k + n, k)\cdot(\ell + m, \ell)$$

   is the form $(p, p + mn)$.

   This again is a simple technical exercise

   $$\begin{aligned}(k + n, k)\ (\ell + m, \ell) &= k\ell + km + k\ell + n\ell,\ k\ell + km + n\ell \\ &\qquad + nm + k\ell) \\ &= (s, s + nm),\end{aligned}$$

   where $s = k\ell + km + k\ell + n\ell$.

   In a similar way the other parts of the rule of signs can be proved.

5. $$ab = ac$$

$$\Rightarrow (a_1, a_2)\cdot(b_1, b_2) \sim (a_1, a_2)\cdot(c_1, c_2)$$

(Note that here we have to use $\sim$ and not $=$. Can you explain why?)

$$\Rightarrow (a_2b_1 + a_1b_2, a_1b_1 + a_2b_2)$$

$$\sim (a_2c_1 + a_1c_2, a_1c_1 + a_2c_2)$$

$$\Rightarrow a_2b_1 + a_1b_2 + a_1c_1 + a_2c_2 = a_1b_1 + a_2b_2 + a_2c_1 + a_1c_2$$

(i) $$\Rightarrow a_1(b_2 + c_1) + a_2(b_1 + c_2) = a_1(b_1 + c_2) + a_2(b_2 + c_1)$$

Since $a \neq 0$, either $a_1 > a_2$ or $a_2 > a_1$. Suppose the former, then the last equation above implies

$$(a_1 - a_2)\cdot(b_2 + c_1) = (a_1 - a_2)\cdot(b_1 + c_2) \qquad \text{(ii)}$$

Now $a_1 - a_2 \in N$. Hence we know (from N) that this equation implies

$$b_2 + c_1 = b_1 + c_2 \qquad \text{(iii)}$$

$$(b_1, b_2) \sim (c_1, c_2)$$

$$b = c$$

But if $a_2 > a_1$, then $a_1 - a_2 \notin N$. So the transition from step (ii) to (iii) is to rely on what we wanted to prove. What would you suggest in this case? The answer is to manipulate differently the equation (i).

We have proved some of the arithmetic results in Z only - all the rest, e.g. associativity and commutativity of addition and multiplication, or $a\cdot b = 0 \Rightarrow a = 0$ or $b = 0$, can be proved similarly, or deduced from results already proved. The object was to exemplify how Z and its arithmetic could be constructed out of N, not to give an exhaustive treatment, which after a while becomes boring.

In our view, the value of these questions to us, as teachers is that one should be aware of the mathematical development; not that one should teach in this way. An approach at the "level" and in the style of Euler or Saunderson would seem to be more appropriate to the classroom. But with a knowledge and appreciation of the formal mathematics, the teacher is aware of the mathematical compromises he is making, and will know which points to emphasize and how to answer students' questions intelligently.

Part III: And so on ...

1. The solution x of $ax = b$ belongs to Z if and only if a divides b. (Note that division in Z has not been defined here - but can be - by methods similar to the above.)

   When a does not divide b, we are in exactly the same position as we were in N with the equation $a + x = b$, when $a > b$. Hence we can play the same game again. Represent x by (a, b). When does (a, b) represent $x \in Z$? what is the definition of equivalent pairs now? Etc. Following exactly similar procedures we can construct Q, the set of rationals.

We still will not have all the positive numbers, because the irrationals are still missing. This is a different story - to be told in another set of worksheets.

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The Negative Numbers

Worksheet: Summary

Part I

The following extract is taken from *Mathematical Thought from Ancient to Modern Times* (1972), a large and comprehensive work by Morris Kline.

In the extract several points in the history of negative numbers are mentioned, including names of mathematicians, their works and some of their arguments.

L. Euler

A. De Morgan

By 1700 all of the familiar members of the system—whole numbers, fractions, irrationals, and negative and complex numbers—were known. However, opposition to the newer types of numbers was expressed throughout the century. Typical are the objections of the English mathematician Baron Francis Masères (1731–1824), a Fellow of Clare College in Cambridge and a member of the Royal Society. Masères, who did write acceptable papers in mathematics and a substantial treatise on the theory of life insurance, published in 1759 his *Dissertation on the Use of the Negative Sign in Algebra*. He shows how to avoid negative numbers (except to indicate the subtraction of a larger quantity from a smaller one), and especially negative roots, by carefully segregating the types of quadratic equations so that those with negative roots are considered separately; and, of course, the negative roots are to be rejected. He does the same with cubics. Then he says of negative roots,

> ...they serve only, as far as I am able to judge, to puzzle the whole doctrine of equations, and to render obscure and mysterious things that are in their own nature exceeding plain and simple.... It were to be wished therefore that negative roots had never been admitted into algebra or were again discarded from it: for if this were done, there is good reason to imagine, the objections which many learned and ingenious men now make to algebraic computations, as being obscure and perplexed with almost unintelligible notions, would be thereby removed; it being certain that Algebra, or universal arithmetic, is, in its own nature, a science no less simple, clear, and capable of demonstration, than geometry.

Certainly negative numbers were not really well understood until modern times. Euler, in the latter half of the eighteenth century, still believed that negative numbers were greater than ∞. He also argued that $(-1)\cdot(-1) = +1$ because the product must be $+1$ or -1 and since $1\cdot(-1) = -1$, then $(-1)\cdot(-1) = +1$. Carnot, the noted French geometer, thought the use of negative numbers led to erroneous conclusions. As late as 1831 Augustus De Morgan (1806–71), professor of mathematics at University College, London, and a famous mathematical logician and contributor to algebra, in his *On the Study and Difficulties of Mathematics*, said, "The imaginary expression $\sqrt{-a}$ and the negative expression $-b$ have this resemblance, that either of them occurring as the solution of a problem indicates some inconsistency or absurdity. As far as real meaning is concerned, both are equally imaginary, since $0 - a$ is as inconceivable as $\sqrt{-a}$."

De Morgan illustrated this by means of a problem. A father is 56; his son is 29. When will the father be twice as old as the son? He solves $56 + x = 2(29 + x)$ and obtains $x = -2$. Thus the result, he says, is absurd. But, he continues, if we change x to $-x$ and solve $56 - x = 2(29-x)$, we get $x = 2$. He concludes that we phrased the original problem wrongly and thus were led to the unacceptable negative answer. De Morgan insisted that it was absurd to consider numbers less than zero.

Questions

1. Complete the information missing in the following summary table.

We shall use the following "notation":

------ to indicate that the information can be found in the extract from Kline.

? to indicate that the extract does not include the information, but it can be found in previous worksheets in this series.

ⓘ to indicate that the extract does not include the information and it is also not to be found in the worksheets.

2. Compare Euler's first referenced argument (i) with the Euler extract brought in the answer sheet: *Contradictions in the use of negative numbers*.

3. Compare Kline's quotation of De Morgan with the two quotations brought in the answer sheet: *The last of the opposition - Frend* (p. 5-6 and 16-17).

4. What can you conclude from your answers to questions 2 and 3?

MATHEMATICIAN	TITLE OF HIS BOOK, PAPER, ETC.	HIS ARGUMENT CONCERNING NEGATIVE NUMBERS
F. Maseres (____-____)	(i) ____________________ ____________________ (ii) ?	(i) "... to avoid negative numbers ... segregating the types of quadratic equations so that ____ ______________________________ ______________________________." (ii) "... when writers of Algebra talk of the negative roots of an equation, they, in fact, jumble two different equations together, ... __ ____?____?____?____ ____?____?____?____."
L. Euler (? - ?)	(i) ?⃝ (ii) ?	(i) "Euler, in the latter half of the eighteenth century, still believed that ______________" (ii) "... (-1)·(-1) = +1 because ______________"
A. De Morgan (____-____)	____________________	"... De Morgan ..., said: ..'either of them [a negative or an imaginary] occurring as the solution of a problem indicates ______________________________' ... De Morgan illustrates this by means of a problem ______________________________"

Part II

In this sequence of worksheets we came across the names of quite a number of people, among them famous mathematicians. We discussed their approach and their position on negative numbers.

In the following we bring,

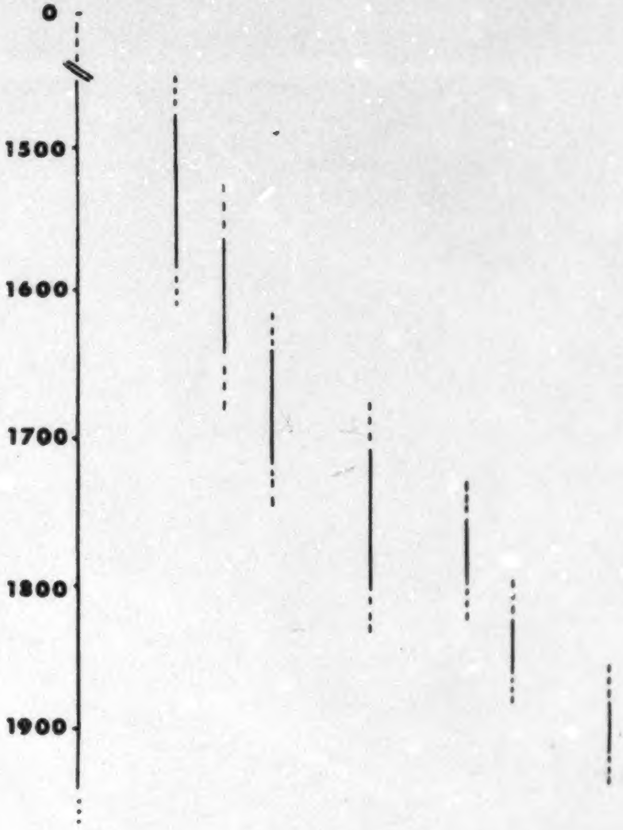
I. Some of these names:

Viète, Descartes, Arnauld, Wallis, Saunderson, Euler, Frend, Maseres, Peacock, De Morgan, Hamilton.

II. A suggested division of the development of the negative number concept into stages:

- *non recognition of negatives,*
- *recognition of negatives as roots of equations,*
- *use of negatives with reservations because of the "contradictions" arising from their use,*
- *the free use of negatives and their entry into textbooks without mathematical definitions,*
- *opposition to negatives,*
- *attempt to give a mathematical foundation of the negatives,*
- *formal mathematical treatment.*

III. A rough chronological table:



Questions

5. i) Identify the various stages in the chronological table.

ii) Insert the name of each mathematician in the appropriate stage.

III) Can you suggest a different division of the development of the negative number concept into stages?

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The Negative Numbers

Answer sheet: Summary

Part I

The extract from *Mathematical Thought from Ancient to Modern Times* (1972) by Kline brings various "facts" from the history of the negative numbers, which we met in previous worksheets and answer sheets.

1. In the following we bring the complete table. Note that, in the cases in which we used the symbol ? to indicate that the information can be found in previous worksheets, we bring here the reference. The symbol Ⓚ is used to denote that the quotation is from the Kline extract.

MATHEMATICIAN	TITLE OF HIS BOOK, PAPER, ETC.	HIS ARGUMENT CONCERNING NEGATIVE NUMBERS
F. Maseres (1731-1824)	(i) *Dissertation on the Use of the Negative Sign in Algebra* (1759)	(i) He shows how to avoid negative numbers (except to indicate the subtraction of a larger quantity from a smaller one), and especially negative roots, by carefully segregating the types of quadratic equations so that those with negative roots are considered separately; and, of course, the negative roots are to be rejected. Ⓚ
	(ii) Appendix to *The Principles of Algebra* (1796) by W. Frend. (See full quotation in the answer sheet: *The Last of the Opposition - W. Frend)*	(ii) when writers of Algebra talk of the negative roots of an equation, they, in ſact, jumble two different equations together. . . Thus, for example, they would ſay, that the quadratick equation $xx + 4x = 320$, has two roots, to wit, the poſitive, or affirmative, root, $+ 16$, and the negative root, $- 20$. But this latter number, 20, is, in truth, the root of a different equation, to wit, of the equation $xx - 4x = 320$.
L. Euler (1707-1783)	(i) (?) (ii) *Elements of Algebra* (1770) (The 1898 edition is quoted in the answer sheet *Euler*)	(i)-(ii) Euler, in the latter half of the eighteenth century, still believed that negative numbers were greater than ∞. He also argued that $(-1)\cdot(-1) = +1$ because the product must be $+1$ or -1 and since $1\cdot(-1) = -1$, then $(-1)\cdot(-1) = +1$. Ⓚ
A. De Morgan (1806-1871)	*On the Studies and Difficulties of Mathematics* (1831- the 1898 edition is quoted in the answer sheet: *The Last of the Opposition - W. Frend)*	The imaginary expression $\sqrt{-a}$ and the negative expression $-b$ have this resemblance, that either of them occurring as the solution of a problem indicates some inconsistency or absurdity. De Morgan illustrated this by means of a problem. A father is 56; his son is 29. When will the father be twice as old as the son? He solves $56 + x = 2(29 + x)$ and obtains $x = -2$. Thus the result, he says, is absurd. But, he continues, if we change x to $-x$ and solve $56 - x = 2(29 - x)$, we get $x = 2$. Ⓚ

2. In the Euler first referenced argument we read that "Euler, in the latter half of the eighteenth century, still believed that negative numbers were greater than ∞." In the answer sheet *Contradictions in the use of negative numbers* we read, in a paragraph from Euler's *Differenzial Rechnung* (1790), the opposite: he destroys the argument of those who concluded that the negatives are greater than infinity.

3. Kline's conclusion of De Morgan's quotations is: "De Morgan insisted that it was absurd to consider numbers less than zero". In the answer sheet *The Last of the Opposition - W. Frend* we read two paragraphs by De Morgan. The first, taken from *A Budget of Paradoxes* is an answer to Maseres' rejection of negatives, in which he says "...the great difficulty of the opponents of algebra lay in want of power or will to see extension of terms...". The second is the same paragraph as that quoted by Kline (with the father-son problem) but with a commentary by De Morgan not brought by Kline, which is: "In either case, however, the result must be interpreted, and a rational answer to the question may be given."

4. The comparisons made in the previous sections lead us to point out how difficult it is to study and/or write history of mathematics, if one wants to be "loyal" to the truth, and to take into account "all the facts". Especially in a monumental work such as that of Kline, it is virtually impossible for the author to check every statement back to its source.

Nevertheless, some advice may be given in order to optimise reliable information, when doing history of mathematics.

We quote from Spalt*(1983):

> Bei der Auseinandersetzung mit mathematischen Texten (und nicht nur solchen!) gilt es, *die goldenen Regeln der Geschichtsschreibung* zu beachten. Deren erste lautet: *Geh' zu den Quellen!* Hier also: Lies die Originaltexte!
> ...
> Nun müssen wir uns der *zweiten goldenen Regel der Geschichtsschreibung* erinnern: *Beachte die Zeitgebundenheit!* Hier also: Bleibe im Textzusammenhang!

Whenever we break one of these rules, which inevitably occurs, we should accept the information with gentle reserve.

We would suggest a further rule, which we hope we have not ourselves broken too often in this series of worksheets. Even after studying the primary source, and making sure that we have understood the context, historical reading or writing involves interpretation, conscious or unconscious. Thus one should be careful not to overstress one's own opinion - that is, to leave a certain measure of self doubt. Historical facts are often not as black and white as they seem. Consider the following.

Kline writes:

> Masères, who did write acceptable papers in mathematics and a substantial treatise on the theory of life insurance,...

Which leaves us with a favourable impression of Maseres work on life insurance.

Now compare De Morgan** on the same work.

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* Spalt D.D., Eine längst fällige, wenngleich unnötige Rehabilitation Cauchys, *Der Mathematik-Unterricht*, 29,4, 1983, p. 60-76.

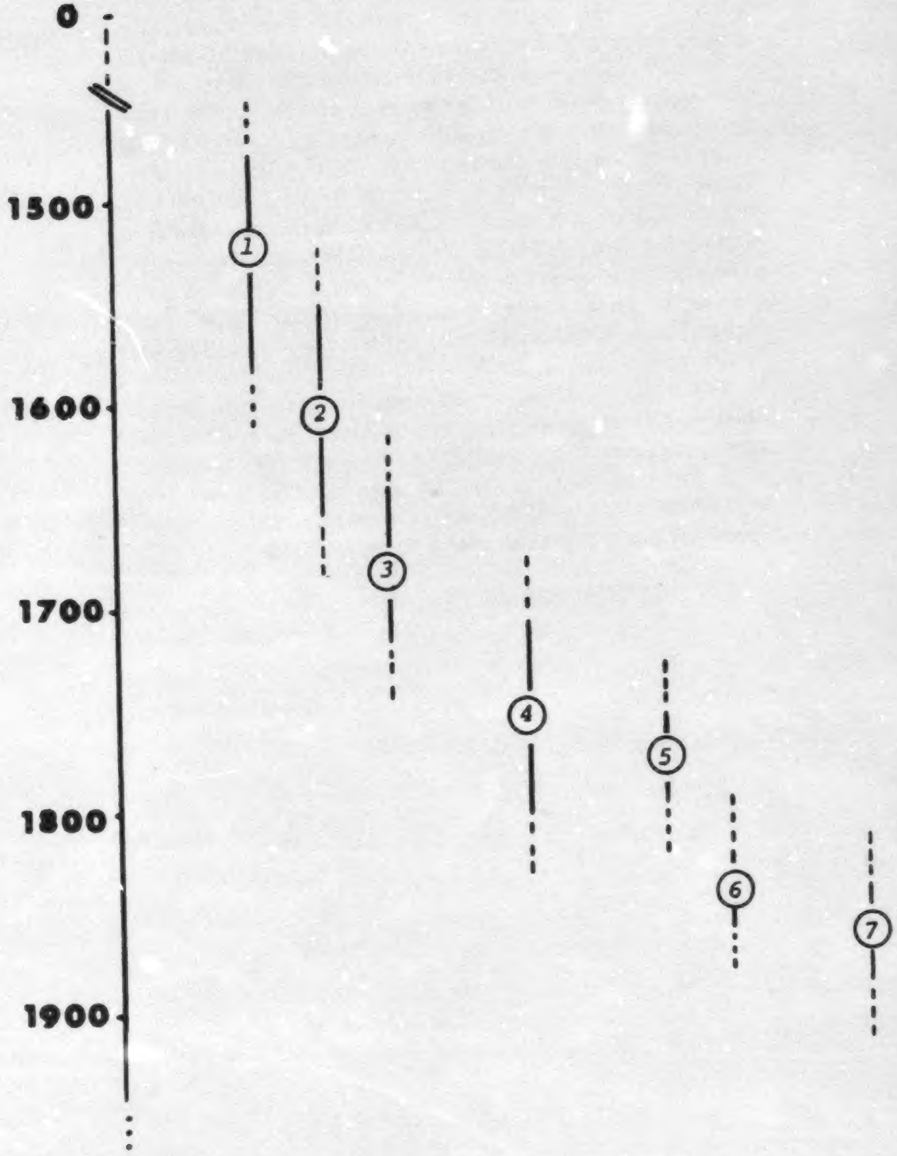
** De Morgan A., *A Budget of Paradoxes*, Vol I. The Open Court Pub., 1915, p. 206

The *Doctrine of Life Annuities*[9] (4to, 726 pages, 1783) is a strange paradox. Its size, the heavy dissertations on the national debt, and the depth of algebra supposed known, put it out of the question as an elementary work, and it is unfitted for the higher student by its elaborate attempt at elementary character, shown in its rejection of forms derived from chances in favor of *the average*, and its exhibition of the separate values of the years of an annuity, as arithmetical illustrations. It is a climax of unsaleability, unreadability, and inutility. For intrinsic nullity of interest, and dilution of little matter with much ink, I can compare this book to nothing but that of Claude de St. Martin, elsewhere mentioned, or the lectures *On the Nature and Properties of Logarithms*, by James Little,[10] Dublin, 1830, 8vo. (254 heavy pages of many words and few symbols), a wonderful weight of weariness.

The stock of this work on annuities, very little diminished, was given by the author to William Frend, who paid warehouse room for it until about 1835, when he consulted me as to its disposal. As no publisher could be found who would take it as a gift, for any purpose of sale, it was consigned, all but a few copies, to a buyer of waste paper.

Part II

5. i-ii)



① *Non recognition of negatives: Viete*

② *Recognition of negatives as roots of equations: Descartes*

③ *Use of negatives with reservations because of the "contradictions" arising from their use: Arnauld, Wallis*

④ *Free use of negatives and their entry into "textbooks" without mathematical definitions: Saunderson, Euler*

⑤ *Opposition to negatives: Frend, Maseres*

⑥ *Attempt to give a mathematical foundation to the negatives: Peacock, De Morgan, Hamilton(?)*

⑦ *Formal mathematical definition: Hamilton(?)*

iii) Note that the above is only a suggested division into stages based on the worksheets in this series. There may be others, especially if we take into account the early use of negatives, for instance, by the Hindus or by Fibonacci. Also other names can be added. Finally, other divisions and/or subdivisions could be made.

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1983